GROUP SYSTEMS, GROUPOIDS, AND MODULI SPACES OF PARABOLIC BUNDLES

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Introduction

Moduli spaces of homomorphisms or more generally twisted homomorphisms from fundamental groups of surfaces to compact connected Lie groups were connected with geometry through their identification with moduli spaces of holomorphic vector bundles [29]. Atiyah and Bott [2] initiated a new approach to the study of these moduli spaces by identifying them with moduli spaces of projectively flat constant central curvature connections on principal bundles over Riemann surfaces, which they analyzed by methods of gauge theory. In particular, they showed that an invariant inner product on the Lie algebra of the Lie group in question induces a natural symplectic structure on a certain smooth open stratum. Although this moduli space is a finite dimensional object, generally a stratified space which is locally semi algebraic [19] but sometimes a manifold, its symplectic structure (on the stratum just mentioned) was obtained by applying the method of symplectic reduction to the action of an infinite dimensional group (the group of gauge transformations) on an infinite dimensional symplectic manifold (the space of all connections on a principal bundle).

This infinite-dimensional approach to moduli spaces has deep roots in quantum field theory [1], but it is nevertheless interesting to try to avoid the technical difficulties of infinite dimensional analysis by using purely finite dimensional methods to construct the symplectic structure and to derive some of its properties. This also allows for arbitrary, not necessarily compact, Lie groups. This program has been carried forward by several authors in the past ten years, with the result being not only technical simplification, but also new insight into the geometry of the moduli spaces, especially into their singularities [17 – 21]. See [22] for a leisurely introduction.

To date, most of the program just described has been worked out only for compact Riemann surfaces without boundary; see however [16]. The purpose of this article is to extend these results and methods to the case of Riemann surfaces with a finite number of punctures or, equivalently, with a finite number of boundary components, corresponding to the study of parabolic vector bundles in the holomorphic category. Specifically, we deal with the results listed below; the references indicate sources for the closed compact case except [16] (see below).

- A description of the symplectic form in terms of the cup product on the cohomology of the fundamental group of the surface in question with values in the Lie algebra [11].
- A proof, using a double complex of Bott and Shulman rather than gauge theory, that the form constructed by using group cohomology is closed [34], thereby allowing for a general Lie group, not necessarily compact.
- A proof, using the Bott-Shulman complex, that the moduli space can be obtained by symplectic reduction from a finite-dimensional symplectic manifold [15, 16, 23, 25].

Some further historical comments may be in order. Regarding the second item above, a proof that the symplectic form is closed, using group cohomology rather than gauge theory, was originally given by Karshon [27]; her proof was reformulated in [34] in terms of the double complex of Bott [6] and Shulman [30]. A partly finite dimensional construction of the moduli space was accomplished earlier by Huebschmann [20] and Jeffrey [24], but in these papers infinite dimensional techniques could not completely

be avoided. A purely finite dimensional construction (for the closed compact case) was announced in [23] and given in [15, 25]. In [26] the Bott-Shulman construction was used to give representatives of all generators of the cohomology ring of certain moduli spaces of representations that are smooth analogues of the moduli space treated in [34].

Passing to the case of punctured surfaces, we note that the symplectic structure on a certain top stratum of the moduli space in this case (see Section 9 below for details about the stratification) was constructed using methods of gauge theory in [4]. A naive attempt to imitate the methods used in the closed compact case [15,23, 25] seems to fail because the concept of fundamental group is too weak to handle peripheral structures; the special case where the fundamental group (of a closed surface) is replaced by an orbifold fundamental group — in the vector bundle case, this corresponds to parabolic bundles with rational weights — has been successfully treated in [16], though.

The principal innovation in this paper is to replace the fundamental group by two more general concepts which enable us to overcome the difficulties with the peripheral structure in general: by that of a *group system* [32] and that of a suitable *fundamental groupoid*. For our purposes, *both* notions do *not* serve for equivalent purposes; rather, the two *complement* each other. Group systems provide the appropriate concept to handle the global structure of the moduli space while the fundamental groupoid turns out to be a crucial tool for a successful treatment of the infinitesimal structure. In fact, a compact orientable topological surface Σ with $n \geq 1$ boundary circles $S_1, \ldots, S_n$ gives rise to a group system $(\pi; \pi_1, \ldots, \pi_n)$ (see Section 1 below for details on this notion) with $\pi = \pi_1(\Sigma), \pi_j = \pi_1(S_j) \cong \mathbf{Z}$, and with a chosen generator z_j of each π_j, referred to henceforth as a *surface group system*. Given a Lie group G, not necessarily compact, and an n-tuple $\mathbf{C} = (C_1, \ldots, C_n)$ of conjugacy classes in G, we denote by $\mathrm{Hom}(\pi, G)_{\mathbf{C}}$ the space of homomorphisms χ from π to G for which the value $\chi(z_j)$ of each generator z_j of lies in C_j, for $1 \leq j \leq n$. Given a nondegenerate invariant symmetric bilinear form on the Lie algebra $\mathfrak{g}$ of G, not necessarily positive definite, we shall construct an *extended moduli space* $\mathcal{M}(\mathcal{P}, G)_{\mathbf{C}}$, that is to say, a smooth symplectic manifold and a hamiltonian G-action and momentum mapping whose reduced space $\mathcal{M}(\mathcal{P}, G)_{\mathbf{C}}//G$ is homeomorphic to the space $\mathrm{Rep}(\pi, G)_{\mathbf{C}}$ of representations, the orbit space for the action of G by conjugation on $\mathrm{Hom}(\pi, G)_{\mathbf{C}}$. The global construction of the space $\mathcal{M}(\mathcal{P}, G)_{\mathbf{C}}$, of its closed 2-form, and of the momentum mapping, involve the surface group system, whereas nondegeneracy of the form is proved by relating the infinitesimal part of structure with Poincaré duality in relative or more precisely *parabolic* cohomology of a certain fundamental groupoid.

Another proof of nondegeneracy is given in a companion paper to this one [13], which uses a metric on the space of parabolic cocycles, leading to a (new?) metric on the moduli space.

For compact G, in the gauge theory setting, the tangent space of an arbitrary point of the top stratum of the moduli space mentioned above in the case of a punctured surface can be identified with the image of compactly supported de Rham cohomology in the usual de Rham cohomology with coefficients in the adjoint bundle, calculated with reference to the operator d_A of covariant derivative with respect to a flat connection A representing the point in question [4]. The cohomology of

"group systems" is the analogue of compactly supported cohomology in the algebraic setting, and the tangent space at a point of the top stratum can be identified with the image of the corresponding group systems cohomology in the usual cohomology. In the last section of this paper, we make this explicit and show that the symplectic structure on the top stratum obtained here by algebraic methods is equivalent to the construction via gauge theory.

Another finite-dimensionalization of the space of flat connections, inspired by lattice gauge theory, was introduced by Fock and Rosly [10].

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1. Group systems

Recall that a *group system* $(\pi;\phi_1,\dots,\phi_n,\pi_1,\dots,\pi_n)$ consists of a group π together with a family of groups π_j and homomorphisms ϕ_j from π_j to π [32]. We shall occasionally refer to the π_j as *peripheral* groups. For any such system, there is a pair of spaces $(X,\cup Y_j)$ such that X and the Y_j are aspherical, $\pi=\pi_1(X),\pi_j=\pi_1(X_j)$, and the maps ϕ_j are induced by inclusion. The *(co)homology of the group system* is that of the pair $(X,\cup Y_j)$. Trotter has given a purely algebraic construction [32]. To introduce notation we reproduce it briefly:

Let R be an arbitrary commutative ring, taken henceforth as ground ring. A *resolution over a system* $(\pi;\phi_1,\dots,\phi_n,\pi_1,\dots,\pi_n)$ is a pair of $R\pi$-complexes (A,B) such that

(1) A is a resolution over π;
(2) B is the direct sum of complexes $B_j=R\pi\otimes_{R\pi_j}A_j$ where A_j is a resolution over π_j and $R\pi$ is considered a right $R\pi_j$-module via the map ϕ_j;
(3) B is a $R\pi$-direct summand of A.

The A_j are referred to as *auxiliary resolutions*; occasionally we shall refer to B as the *peripheral part* of the resolution. Given a resolution (A,B) over a system $(\pi;\phi_1,\dots,\phi_n,\pi_1,\dots,\pi_n)$, the exact sequence

$$0\longrightarrow B\longrightarrow A\longrightarrow A/B\longrightarrow 0 \tag{1.1}$$

of $R\pi$-modules splits, and the homology $\mathrm{H}_*(\{\phi_j\},\cdot)$ and cohomology $\mathrm{H}^*(\{\phi_j\},\cdot)$ of the system are defined from the $R\pi$-complex A/B, which plays the role of a "relative" resolution. In particular, for every $R\pi$-module M, the exact sequence (1.1) gives rise to natural long exact sequences in homology and cohomology of the kind

$$\ldots\longrightarrow \mathrm{H}_k(\{\pi_j\},M)\longrightarrow \mathrm{H}_k(\pi,M)\longrightarrow \mathrm{H}_k(\{\phi_j\},M)\longrightarrow \mathrm{H}_{k-1}(\{\pi_j\},M)\longrightarrow\ldots \tag{1.2}$$

and

$$\ldots\longleftarrow \mathrm{H}^{k+1}(\{\phi_j\},M)\longleftarrow \mathrm{H}^k(\{\pi_j\},M)\longleftarrow \mathrm{H}^k(\pi,M)\longleftarrow \mathrm{H}^k(\{\phi_j\},M)\longleftarrow\ldots\ . \tag{1.3}$$

Here $\mathrm{H}_k(\{\phi_j\},M)$ and $\mathrm{H}^k(\{\phi_j\},M)$ are just the direct sums of the homology and cohomology groups $\mathrm{H}_k(\pi_j,M)$ and $\mathrm{H}^k(\pi_j,M)$, respectively.

Given a group system $(\pi;\phi_1,\dots,\phi_n,\pi_1,\dots,\pi_n)$ where the ϕ_j are inclusions of subgroups we shall henceforth suppress $\phi_1,\dots,\phi_n$ in notation and simply write $(\pi;\pi_1,\dots,\pi_n)$ and, likewise, we shall write $\mathrm{H}_*(\pi,\{\pi_j\};\cdot)$ and $\mathrm{H}^*(\pi,\{\pi_j\};\cdot)$ for $\mathrm{H}_*(\{\phi_j\},\cdot)$ and $\mathrm{H}^*(\{\phi_j\},\cdot)$, respectively.

The explicit construction of a resolution of a system can concisely be handled by means of a corresponding *fundamental groupoid*. We explain this for surface group systems in the next section.

2. Surface group systems

Let Σ be a compact orientable topological surface with boundary $\partial\Sigma$ consisting of n circles $S_1, \dots, S_n$; we suppose that when the genus ℓ of Σ is zero there are $n \geq 3$ boundary circles. This surface gives rise to a group system $(\pi; \pi_1, \dots, \pi_n)$ with $\pi = \pi_1(\Sigma), \pi_j = \pi_1(S_j) \cong \mathbf{Z}$, referred to henceforth as a *surface group system.* When the boundary $\partial\Sigma$ is non-empty the group π is free non-abelian; yet it is convenient to use the presentation

$$\mathcal{P} = \langle x_1, y_1, \dots, x_\ell, y_\ell, z_1, \dots, z_n; r\rangle, \tag{2.1}$$

where

$$r = \Pi[x_j, y_j]z_1 \dots z_n.$$

The Reidemeister-Fox calculus, applied to the presentation $\mathcal{P}$, yields the free resolution

$$\mathbf{R}(\mathcal{P})\colon R_2(\mathcal{P}) \xrightarrow{\partial_2} R_1(\mathcal{P}) \xrightarrow{\partial_1} R\pi \tag{2.2}$$

of R in the category of left $R\pi$-modules. Here

$$R_2(\mathcal{P}) = R\pi[r], \quad R_1(\mathcal{P}) = R\pi[x_1, y_1, \dots, x_\ell, y_\ell, z_1, \dots, z_n],$$

and the boundary operators ∂_j are given by the formulas

$$\begin{aligned}
\partial_1[x_i] &= (x_i - 1)\\
\partial_1[y_i] &= (y_i - 1)\\
\partial_1[z_j] &= (z_j - 1)\\
\partial_2[r] &= \sum \frac{\partial r}{\partial x_i}[x_i] + \sum \frac{\partial r}{\partial y_i}[y_i] + \sum \frac{\partial r}{\partial z_j}[z_j].
\end{aligned}$$

Here and henceforth we decorate the free generators of the modules coming into play in the resolution by square brackets, to distinguish them from the corresponding elements of the group π etc. The chain complex arising from (2.2) which calculates the absolute homology of π with values in R comes down to

$$R[r] \xrightarrow{\overline{\partial}_2} R[x_1, y_1, \dots, x_\ell, y_\ell, z_1, \dots, z_n] \xrightarrow{0} R, \quad \overline{\partial}_2[r] = [z_1] + \dots + [z_n]. \tag{2.3}$$

A resolution of the group system is concisely handled by means of the following groupoid the full force of which will be exploited only in Section 8, though. Pick a base point p_0 not on the boundary and, moreover, for each boundary component S_j, pick a base point p_j. This determines the subgroupoid $\widetilde{\pi} = \Pi(\Sigma; p_0, p_1, \dots, p_n)$ of the fundamental groupoid of Σ consisting of homotopy classes of paths in Σ with endpoints contained in the set $\{p_0, p_1, \dots, p_n\}$. To obtain a presentation of it we decompose Σ into cells as follows, where we do not distinguish in notation between the chosen edge paths and their homotopy classes relative to their end points: Let $x_1, y_1, \dots, x_\ell, y_\ell$ be closed paths which (i) do not meet the boundary, (ii) have p_0 as starting point, and (iii) yield the generators respectively $x_1, y_1, \dots, x_\ell, y_\ell$ of the fundamental group $\pi = \pi_1(\Sigma, p_0)$; for $j = 1, \dots, n$, let a_j be the boundary path of

the j'th boundary circle, having p_j as starting point, and let γ_j be a path from p_0 to p_j. When we cut Σ along these 1-cells we obtain a disk D whose boundary yields the defining relation of $\widetilde{\pi} = \Pi(\Sigma; p_0, p_1, \ldots, p_n)$. The resulting presentation of $\widetilde{\pi}$ looks like

$$\widetilde{\mathcal{P}} = \langle x_1, y_1, \ldots, x_\ell, y_\ell, a_1, \ldots, a_n, \gamma_1, \ldots, \gamma_n; \widetilde{r}\rangle, \tag{2.4}$$

where the relator now reads

$$\widetilde{r} = \Pi[x_j, y_j]\Pi\gamma_j a_j \gamma_j^{-1}.$$

This is consistent with the presentation (2.1) of the fundamental group if we identify each generator z_j with $\gamma_j a_j \gamma_j^{-1}$. Reading the boundary $\widetilde{r}$ counterclockwise around D determines an orientation of D and hence of Σ in the usual way.

The Reidemeister-Fox calculus, applied to $\widetilde{\mathcal{P}}$, yields the free resolution of R

$$\mathbf{R}(\widetilde{\mathcal{P}})\colon R_2(\widetilde{\mathcal{P}}) \xrightarrow{\partial_2} R_1(\widetilde{\mathcal{P}}) \xrightarrow{\partial_1} R_0(\widetilde{\mathcal{P}}) = R\pi[p_0, \ldots, p_n] \tag{2.5}$$

in the category of left $R\pi$-modules. Here

$$R_2(\widetilde{\mathcal{P}}) = R\pi[\widetilde{r}], \quad R_1(\widetilde{\mathcal{P}}) = R\pi[x_1, y_1, \ldots, x_\ell, y_\ell, a_1, \ldots, a_n, \gamma_1, \ldots, \gamma_n],$$

and

$$\begin{aligned}
\partial_1[x_i] &= (x_i - 1)[p_0],\\
\partial_1[y_i] &= (y_i - 1)[p_0],\\
\partial_1[a_j] &= (z_j - 1)[p_j],\\
\partial_1[\gamma_j] &= [p_j] - [p_0],\\
\partial_2[\widetilde{r}] &= \sum \frac{\partial r}{\partial x_i}[x_i] + \sum \frac{\partial r}{\partial y_i}[y_i] + \sum \frac{\partial r}{\partial z_j}[a_j] + \sum \frac{\partial r}{\partial z_j}(1 - z_j)[\gamma_j].
\end{aligned} \tag{2.6}$$

Notice that this amounts to a concise description of the cellular chains of the universal cover $\widetilde{\Sigma}$ of Σ whence it is manifestly a free resolution. This description of the chains of the universal cover will be exploited in Section 10 below. Alternatively, observe that dividing out the contractible $R\pi$-subcomplex generated by the $[\gamma_j]$ and $[p_j] - [p_0]$ transforms (2.5) into the free resolution (2.2) above The resulting chain complex calculating the homology of π with values in R amounts to

$$R[r] \xrightarrow{\overline{\partial}_2} R[x_1, y_1, \ldots, x_\ell, y_\ell, a_1, \ldots, a_n, \gamma_1, \ldots, \gamma_n] \xrightarrow{\overline{\partial}_1} R \tag{2.7}$$

where

$$\begin{aligned}
\overline{\partial}_2[\widetilde{r}] &= [a_1] + \cdots + [a_n], &&\\
\overline{\partial}_1[x_i] = \overline{\partial}_1[y_i] &= 0, && 1 \le i \le \ell,\\
\overline{\partial}_1[a_j] &= 0, && 1 \le j \le n.\\
\overline{\partial}_1[\gamma_j] &= [p_j] - [p_0], && 1 \le j \le n.
\end{aligned}$$

It equals that of cellular chains of Σ, when we identify the disk D with $[\widetilde{r}]$. Notice that when $n \geq 1$, the orientation of D is determined by the boundary relation

$$\overline{\partial} D = [a_1] + \cdots + [a_n].$$

To see that (2.5) yields a resolution over our group system, for $j = 1, \ldots, n$, let A_j be the small resolution of R over the free cyclic group π_j determined by the choice of generator a_j, and let $B_j = R\pi \otimes_{R\pi_j} A_j$; explicitly:

$$B_j \colon R\pi[a_j] \xrightarrow{\partial} R\pi[p_j], \quad \partial[a_j] = (z_j - 1)[p_j].$$

The resolution (2.5) plainly contains the $R\pi$-complex $B = \oplus_{j=1,\ldots,n} B_j$ as a direct summand. Hence $(\mathbf{R}(\widetilde{\mathcal{P}}), B)$ is a resolution over our group system. Notice that for $\ell = 0$ and $n = 1$ the construction does *not* yield a resolution over the corresponding group system.

By construction, the quotient complex $\mathbf{R}(\widetilde{\mathcal{P}}, \{\pi_j\})$ calculating the (co)homology of our surface system arises from (2.5) by dividing out the subcomplex generated by the $[a_j]$ and $[p_j]$, for $1 \leq j \leq n$. Thus it looks like

$$\mathbf{R}(\widetilde{\mathcal{P}}, \{\pi_j\}) \colon R\pi[\widetilde{r}] \xrightarrow{\partial_2} R\pi[x_1, y_1, \ldots, x_\ell, y_\ell, \gamma_1, \ldots, \gamma_n] \xrightarrow{\partial_1} R\pi; \tag{2.8}$$

its boundary operators ∂_j are given by the formulas

$$\begin{aligned} \partial_1[x_i] &= (x_i - 1) \\ \partial_1[y_i] &= (y_i - 1) \\ \partial_1[\gamma_j] &= 1 \\ \partial_2[\widetilde{r}] &= \sum \frac{\partial r}{\partial x_i}[x_i] + \sum \frac{\partial r}{\partial y_i}[y_i] + \sum \frac{\partial r}{\partial z_j}(1 - z_j)[\gamma_j]. \end{aligned} \tag{2.9}$$

In particular, the chain complex calculating the homology of the system with values in R amounts to

$$R[r] \xrightarrow{\overline{\partial}_2} R[x_1, y_1, \ldots, x_\ell, y_\ell, \gamma_1, \ldots, \gamma_n] \xrightarrow{\overline{\partial}_1} R \tag{2.10}$$

where

$$\begin{aligned} \overline{\partial}_2[\widetilde{r}] &= 0 \\ \overline{\partial}_1[x_i] = \overline{\partial}_1[y_i] &= 0, \quad 1 \leq i \leq \ell, \\ \overline{\partial}_1[\gamma_j] &= 1, \qquad\quad 1 \leq j \leq n. \end{aligned}$$

Thus the 2-chain

$$b = \widetilde{r} \tag{2.11}$$

is a relative 2-cycle, and $\mathrm{H}_2(\pi, \{\pi_j\}; R)$ is isomorphic to R, generated by the class κ of b.

A comparison map from $\mathbf{R}(\mathcal{P})$ to $\mathbf{R}(\widetilde{\mathcal{P}}, \{\pi_j\})$ inducing the cohomology map from $\mathrm{H}^*(\pi, \{\pi_j\}; \cdot)$ to $\mathrm{H}^*(\pi, \cdot)$ occurring in (1.3) above is given by the chain map

$$\begin{array}{lccccc} \mathbf{R}(\mathcal{P}): & R\pi[r] & \xrightarrow{\partial_2} & R\pi[x_1, y_1, \dots, x_\ell, y_\ell, z_1, \dots, z_n] & \xrightarrow{\partial_1} & R\pi \\ & \downarrow & & \downarrow & & \downarrow \\ \mathbf{R}(\widetilde{\mathcal{P}}, \{\pi_j\}): & R\pi[\widetilde{r}] & \xrightarrow{\partial_2} & R\pi[x_1, y_1, \dots, x_\ell, y_\ell, \gamma_1, \dots, \gamma_n] & \xrightarrow{\partial_1} & R\pi \end{array} \tag{2.12}$$

which identifies the elements denoted by the same symbols in the top and bottom row and sends $[r]$ to $[\widetilde{r}]$ and $[z_j]$ to $(z_j - 1)[\gamma_j]$, for $1 \le j \le n$. In particular, under the induced chain map from (2.3) to (2.10), the boundary value $[z_1] + \dots + [z_n]$ of $[r]$ in (2.3) goes to zero.

3. Poincaré duality for surface group systems

The surface group system $(\pi; \pi_1, \dots, \pi_n)$ is a *two-dimensional Poincaré duality system* over R, that is, a PD^2-*pair* in the terminology of Bieri-Eckmann [3], having fundamental class $\kappa \in \mathrm{H}_2(\pi, \{\pi_j\}; R)$, so that, for every $R\pi$-module M, cap product with κ yields natural isomorphisms

$$\cap\kappa \colon \mathrm{H}^*(\pi, M) \to \mathrm{H}_{2-*}(\pi, \{\pi_j\}; M), \quad \cap\kappa \colon \mathrm{H}^*(\pi, \{\pi_j\}; M) \to \mathrm{H}_{2-*}(\pi, M), \tag{3.1}$$

cf. [3], and these in fact fit into a commutative diagram

$$\begin{array}{ccccccc} \longrightarrow & \mathrm{H}^*(\pi, \{\pi_j\}; M) & \longrightarrow & \mathrm{H}^*(\pi, M) & \longrightarrow & \mathrm{H}^*(\{\pi_j\}, M) & \longrightarrow \\ & \downarrow{\scriptstyle \cap\kappa} & & \downarrow{\scriptstyle \cap\kappa} & & \downarrow{\scriptstyle \cap\partial\kappa} & \\ \longrightarrow & \mathrm{H}_{2-*}(\pi, M) & \longrightarrow & \mathrm{H}_{2-*}(\pi, \{\pi_j\}; M) & \longrightarrow & \mathrm{H}_{1-*}(\{\pi_j\}, M) & \longrightarrow \end{array}$$

whose horizontal sequences are the corresponding long exact homology and cohomology sequences of the group system. In particular, $\mathrm{H}^2(\pi, \{\pi_j\}; R)$ is also just a copy of R. Geometrically this duality is exactly that of the surface Σ with boundary $\partial\Sigma$ with coefficients determined by M, viewed as a local system, that is,

$$\cap e \colon \mathrm{H}^*(\Sigma, M) \to \mathrm{H}_{2-*}(\Sigma, \partial\Sigma; M), \quad \cap e \colon \mathrm{H}^*(\Sigma, \partial\Sigma; M) \to \mathrm{H}_{2-*}(\Sigma, M), \tag{3.2}$$

where $e \in \mathrm{H}_2(\Sigma, \partial\Sigma; R)$ refers to the orientation class.

Let $R = \mathbb{R}$, the reals, let V be a finite dimensional real vector space with a symmetric bilinear form $\cdot$, and suppose V endowed with a structure of $\mathbb{R}\pi$-module preserving the given symmetric bilinear form (i.e. the action of π preserves the form). Via the multiplicative structure of the cohomology of a group system — this amounts of course to the multiplicative structure of the cohomology of the pair $(\Sigma, \partial\Sigma)$ — the symmetric bilinear form $\cdot$ induces a pairing

$$\mathrm{H}^1(\pi, \{\pi_j\}; V) \otimes \mathrm{H}^1(\pi, V) \to \mathrm{H}^2(\pi, \{\pi_j\}; \mathbb{R}) \tag{3.3}$$

which, combined with

$$\cap\kappa \colon \mathrm{H}^2(\pi, \{\pi_j\}; \mathbb{R}) \to \mathrm{H}_0(\pi, \mathbb{R}) = \mathbb{R},$$

yields a bilinear pairing

$$\mathrm{H}^1(\pi,\{\pi_j\};V)\otimes \mathrm{H}^1(\pi,V)\longrightarrow \mathbb{R}; \tag{3.4}$$

Poincaré duality in the cohomology of the group system implies that the pairing (3.4) is nondegenerate provided that $\cdot$ is nondegenerate. The multiplicative structure has been made explicit in [32] for systems with a single peripheral subgroup. By means of an appropriate groupoid we shall make explicit the multiplicative structure in Section 8 below in the general case.

Next we write $\mathrm{H}^1_{\mathrm{par}}(\pi,\{\pi_j\};V)$ for the image of $\mathrm{H}^1(\pi,\{\pi_j\};V)$ in $\mathrm{H}^1(\pi,V)$ under the canonical map, cf. (1.3), and we refer to it as (first) *parabolic cohomology*, with values in V. Parabolic cohomology classes are represented by *parabolic* 1-cocycles, that is, by 1-cocycles $\zeta\colon\pi\to V$ having the property that, for every z_j, $1\le j\le n$, there is an element X_j in V such that

$$\zeta(z_j)=z_jX_j-X_j. \tag{3.5}$$

We denote the space of parabolic 1-cocycles by $Z^1_{\mathrm{par}}(\pi,\{\pi_j\};V)$. Parabolic 1-cocycles and parabolic cohomology have been introduced by A. Weil [33], for arbitrary finitely generated planar discontinuous groups, and he noticed that, for such a group with only elliptic and hyperbolic generators (which, in the present description, amounts to imposing the additional relations saying that every generator of the kind z_j has finite order) every 1-cocycle is parabolic since the cohomology of a finite group with coefficients in a real vector space is trivial.

Using the top row of the commutative diagram after (3.1), we have an exact sequence

$$0\longrightarrow \mathrm{Ker}(j)\longrightarrow \mathrm{H}^1(\pi,\{\pi_j\};V)\xrightarrow{j}\mathrm{H}^1(\pi,V)\longrightarrow \mathrm{Coker}(j)\longrightarrow 0$$

and the restriction of (3.4) to $\mathrm{Ker}(j)\otimes\mathrm{Im}(j)$ is zero where $\mathrm{Im}(j)$ refers to the image of j in $\mathrm{H}^1(\pi,V)$. This implies that the pairing (3.4) yields a pairing

$$\big(\mathrm{H}^1(\pi,\{\pi_j\};V)/\mathrm{Ker}(j)\big)\otimes \mathrm{Im}(j)\longrightarrow \mathbb{R}.$$

However j induces an isomorphism from $\mathrm{H}^1(\pi,\{\pi_j\};V)/\mathrm{Ker}(j)$ onto $\mathrm{Im}(j)$ which equals $\mathrm{H}^1_{\mathrm{par}}(\pi,\{\pi_j\};V)$. Hence the pairing (3.4) induces a skew-symmetric bilinear pairing

$$\omega_V\colon \mathrm{H}^1_{\mathrm{par}}(\pi,\{\pi_j\};V)\otimes \mathrm{H}^1_{\mathrm{par}}(\pi,\{\pi_j\};V)\longrightarrow \mathbb{R}. \tag{3.6}$$

When $\cdot$ is nondegenerate, so is (3.4), and $\mathrm{Im}(j)$ equals the annihilator in $\mathrm{H}^1(\pi,V)$ of $\mathrm{Ker}(j)$. Hence (3.6) is nondegenerate, that is, a symplectic structure on the vector space $\mathrm{H}^1_{\mathrm{par}}(\pi,\{\pi_j\};V)$, provided that $\cdot$ is nondegenerate. This pairing will be explicitly calculated in Lemma 8.4 below.

4. Representation spaces of surface group systems

Write F for the free group on the generators in $\mathcal{P}$. Let G be a Lie group, not necessarily compact, write $\mathfrak{g}$ for its Lie algebra, and let $\mathbf{C} = \{C_1, \dots, C_n\}$ be an n-tuple of conjugacy classes in G.

Let $\phi \in \mathrm{Hom}(F, G)$, and suppose that $\phi(r)$ lies in the centre of G. Then the composite of ϕ with the adjoint representation of G induces a structure of a (left) π-module on $\mathfrak{g}$, and we write $\mathfrak{g}_\phi$ for $\mathfrak{g}$, viewed as as π-module in this way. We shall continue to take as ground ring R the reals $\mathbb{R}$. Application of the functor $\mathrm{Hom}_{\mathbb{R}\pi}(\cdot, \mathfrak{g}_\phi)$ to the free resolution $\mathbf{R}(\mathcal{P})$ yields the chain complex

$$\mathbf{C}(\mathcal{P}, \mathfrak{g}_\phi)\colon \mathrm{C}^0(\mathcal{P}, \mathfrak{g}_\phi) \xrightarrow{\delta^0_\phi} \mathrm{C}^1(\mathcal{P}, \mathfrak{g}_\phi) \xrightarrow{\delta^1_\phi} \mathrm{C}^2(\mathcal{P}, \mathfrak{g}_\phi), \tag{4.1}$$

cf. [15] (4.1), computing the group cohomology $\mathrm{H}^*(\pi, \mathfrak{g}_\phi)$; we recall that there are canonical isomorphisms

$$\mathrm{C}^0(\mathcal{P}, \mathfrak{g}_\phi) \cong \mathfrak{g}, \quad \mathrm{C}^1(\mathcal{P}, \mathfrak{g}_\phi) \cong \mathfrak{g}^{2\ell+n}, \quad \mathrm{C}^2(\mathcal{P}, \mathfrak{g}_\phi) \cong \mathfrak{g}.$$

To recall the geometric significance of this chain complex, denote by α_ϕ the smooth map from G to $\mathrm{Hom}(F, G)$ which assigns $x\phi x^{-1}$ to $x \in G$, write r for the smooth map from $\mathrm{Hom}(F, G)$ to G induced by the relator r so that the pre-image of the neutral element e of G equals the space $\mathrm{Hom}(\pi, G)$, and write $R_\phi\colon \mathfrak{g}^{2\ell+n} \to \mathrm{T}_\phi \mathrm{Hom}(F, G)$ and $R_{r(\phi)}\colon \mathfrak{g} \to \mathrm{T}_{r(\phi)} G$ for the corresponding operations of right translation. The tangent maps $\mathrm{T}_e \alpha_\phi$ and $\mathrm{T}_\phi r$ make commutative the diagram

$$\begin{tikzcd}
\mathrm{T}_e G \arrow[r, "\mathrm{T}_e\alpha_\phi"] & \mathrm{T}_\phi \mathrm{Hom}(F, G) \arrow[r, "\mathrm{T}_\phi r"] & \mathrm{T}_{r(\phi)} G \\
\mathfrak{g} \arrow[u, "\mathrm{Id}"] \arrow[r, "\delta^0_\phi"'] & \mathfrak{g}^{2\ell+n} \arrow[u, "\mathrm{R}_\phi"] \arrow[r, "\delta^1_\phi"'] & \mathfrak{g}, \arrow[u, "\mathrm{R}_{r(\phi)}"]
\end{tikzcd} \tag{4.2}$$

cf. [15] (4.2). The commutativity of this diagram shows at once that right translation identifies the kernel of the derivative $\mathrm{T}_\phi r$ with the kernel of the coboundary operator δ^1_ϕ from $\mathrm{C}^1(\mathcal{P}, \mathfrak{g}_\phi)$ to $\mathrm{C}^2(\mathcal{P}, \mathfrak{g}_\phi)$, that is, with the vector space $\mathrm{Z}^1(\pi, \mathfrak{g}_\phi)$ of $\mathfrak{g}_\phi$-valued 1-cocycles of π; this space does *not* depend on a specific presentation $\mathcal{P}$, whence the notation. We note that $\mathrm{C}^1(\mathcal{P}, \mathfrak{g}_\phi) = Z^1(F, \mathfrak{g}_\phi)$, the space of $\mathfrak{g}_\phi$-valued 1-cocycles for F.

For each j, $1 \le j \le n$, write F_j for the subgroup of F generated by z_j. We then have two group systems $(F; F_1, \dots, F_n)$ and $(\pi; \pi_1, \dots, \pi_n)$, together with the obvious morphism of group systems from the former to the latter. Notice that, for each j, the corresponding homomorphism from F_j to π_j is an isomorphism but we prefer to maintain a distinction in notation between F_j and π_j. Extending notation introduced earlier, we denote by $\mathrm{Hom}(F, G)_{\mathbf{C}}$ the space of homomorphisms ϕ from F to G for which the value $\phi(z_j)$ of each generator z_j lies in C_j, for $1 \le j \le n$. The choice of generators induces a decomposition

$$\mathrm{Hom}(F, G)_{\mathbf{C}} \cong G^{2\ell} \times C_1 \times \cdots \times C_n.$$

Abusing notation, we denote the restriction of r to $\mathrm{Hom}(F,G)_{\mathbf{C}}$ by r as well, so that the pre-image $r^{-1}(e) \subseteq \mathrm{Hom}(F,G)_{\mathbf{C}}$ of the neutral element e of G equals the space $\mathrm{Hom}(\pi,G)_{\mathbf{C}}$.

We now suppose that our chosen $\phi \in \mathrm{Hom}(F,G)$ lies in $\mathrm{Hom}(F,G)_{\mathbf{C}}$, viewed as a subspace of $\mathrm{Hom}(F,G)$. For $j=1,\dots,n$, denote by h_j the image in $\mathfrak g$ of the linear endomorphism given by $\mathrm{Ad}(\phi(z_j)) - \mathrm{Id}$, so that there results the exact sequence

$$0 \longrightarrow \mathfrak g_j \longrightarrow \mathfrak g \longrightarrow h_j \longrightarrow 0 \tag{4.3}$$

of vector spaces, where $\mathfrak g_j$ denotes the Lie algebra of the stabilizer of $\phi(z_j)$; notice that h_j is the tangent space of the conjugacy class C_j.

The following two observations will be crucial.

Proposition 4.4. *The values of the operator δ^0_ϕ in (4.2) lie in $\mathfrak g^{2\ell} \times h_1 \times \dots \times h_n$, viewed as a subspace of $\mathrm{C}^1(\mathcal P, \mathfrak g_\phi) \cong \mathfrak g^{2\ell} \times \mathfrak g^n$, and the first cohomology group of the resulting complex*

$$\mathbf{C}_{\mathrm{par}}(\mathcal P, \mathfrak g_\phi)\colon \mathfrak g \xrightarrow{\delta^0_\phi} \mathfrak g^{2\ell} \times h_1 \times \dots \times h_n \xrightarrow{\delta^1_\phi} \mathfrak g \tag{4.4.1}$$

equals $\mathrm{H}^1_{\mathrm{par}}(\pi, \{\pi_j\}; \mathfrak g_\phi)$.

Proof. Let

$$\Phi = (\mathrm{Ad}(z_1) - \mathrm{Id}, \dots, \mathrm{Ad}(z_n) - \mathrm{Id})\colon \mathfrak g^n \longrightarrow \mathfrak g^n.$$

Application of the functor $\mathrm{Hom}_{\mathbb R\pi}(\cdot, \mathfrak g_\phi)$ to (2.12) yields the cochain map

$$\begin{array}{cccccc}
\mathbf{C}(\mathcal P, \mathfrak g_\phi)\colon & \mathfrak g & \xrightarrow{\delta^0_\phi} & \mathfrak g^{2\ell} \times \mathfrak g^n & \xrightarrow{\delta^1_\phi} & \mathfrak g \\
 & {\scriptstyle \mathrm{Id}}\uparrow & & {\scriptstyle (\mathrm{Id},\Phi)}\uparrow & & {\scriptstyle \mathrm{Id}}\uparrow \\
\mathbf{C}(\widetilde{\mathcal P}, \{\pi_j\}; \mathfrak g_\phi)\colon & \mathfrak g & \xrightarrow[\delta^0_\phi]{} & \mathfrak g^{2\ell} \times \mathfrak g^n & \xrightarrow[\delta^1_\phi]{} & \mathfrak g
\end{array}$$

where the notation δ^0_ϕ and δ^1_ϕ is slightly abused. It is obvious that this chain map factors through $\mathbf{C}_{\mathrm{par}}(\mathcal P, \mathfrak g_\phi)$. A little thought reveals that this implies the assertion. □

Proposition 4.5. *The tangent maps $\mathrm{T}_e\alpha_\phi$ and $\mathrm{T}_\phi r$ make commutative the diagram*

$$\begin{array}{cccccc}
 & \mathrm{T}_e G & \xrightarrow{\mathrm{T}_e\alpha_\phi} & \mathrm{T}_\phi\left(\mathrm{Hom}(F,G)_{\mathbf{C}}\right) & \xrightarrow{\mathrm{T}_\phi r} & \mathrm{T}_{r(\phi)}G \\
 & {\scriptstyle \mathrm{Id}}\uparrow & & {\scriptstyle \mathrm{R}_\phi}\uparrow & & {\scriptstyle \mathrm{R}_{r(\phi)}}\uparrow \\
\mathbf{C}_{\mathrm{par}}(\mathcal P, \mathfrak g_\phi)\colon & \mathfrak g & \xrightarrow[\delta^0_\phi]{} & \mathfrak g^{2\ell} \times h_1 \times \dots \times h_n & \xrightarrow[\delta^1_\phi]{} & \mathfrak g,
\end{array} \tag{4.5.1}$$

having its vertical arrows isomorphisms of vector spaces.

Proof. In fact, the diagram (4.2) restricts to the diagram (4.5.1). □

It is manifest that the kernel of the operator δ^1_ϕ in $\mathbf{C}_{\mathrm{par}}(\mathcal P, \mathfrak g_\phi)$ (cf. (4.4.1)) coincides with the space $Z^1_{\mathrm{par}}(\pi, \{\pi_j\}; \mathfrak g_\phi)$ of parabolic 1-cocycles with values in $\mathfrak g_\phi$.

5. The extended moduli space

Let $\cdot$ be an invariant symmetric bilinear form on $\mathfrak{g}$, not necessarily positive definite or even nondegenerate. The additional hypothesis of nondegeneracy will be exploited only in Section 8 – 10 below. As in [15], for a group Π, we denote by $(C_*(\Pi), \partial)$ the chain complex of its nonhomogeneous reduced normalized bar resolution over the ground ring R. When the relators $z_1, \dots, z_n$ are added to (2.1) we obtain the presentation

$$\widehat{\mathcal{P}} = \langle x_1, y_1, \dots, x_\ell, y_\ell, z_1, \dots, z_n; r, z_1, \dots, z_n \rangle \tag{5.1}$$

of the fundamental group $\widehat{\pi} = \pi_1(\widehat{\Sigma})$ of the *closed* surface $\widehat{\Sigma}$ resulting from capping of the n boundaries. Let F be the free group on the generators of $\mathcal{P}$; notice that the generators of the latter coincide with those of $\widehat{\mathcal{P}}$. We apply a variant of the construction in [15] to the presentation $\widehat{\mathcal{P}}$: Let O be the open G-invariant subset of the Lie algebra $\mathfrak{g}$ of G where the exponential mapping is regular. Define the space $\mathcal{H}(\mathcal{P}, G)_{\mathbf{C}}$ by means of the pull back square

$$\begin{tikzcd}
\mathcal{H}(\mathcal{P}, G)_{\mathbf{C}} \arrow[r, "{(\widehat{r}, \overline{z}_1, \dots, \overline{z}_n)}"] \arrow[d, "\eta"'] & O \times C_1 \times \cdots \times C_n \arrow[d, "\exp \times \mathrm{Id} \times \cdots \times \mathrm{Id}"] \\
\mathrm{Hom}(F, G)_{\mathbf{C}} \arrow[r, "{(r, z_1, \dots, z_n)}"'] & G \times C_1 \times \cdots \times C_n,
\end{tikzcd} \tag{5.2}$$

where $\widehat{r}$ and $\overline{z}_1, \dots, \overline{z}_n$ denote the induced maps. The space $\mathcal{H}(\mathcal{P}, G)_{\mathbf{C}}$ is manifestly a smooth manifold.

Let c be an absolute 2-chain of F which represents a 2-cycle for the group system $(\pi; \pi_1, \dots, \pi_n)$. Its image in the 2-chains of the fundamental group $\widehat{\pi}$ of the *closed* (!) surface $\widehat{\Sigma}$ is then closed. Write $\widehat{\kappa} \in \mathrm{H}_2(\widehat{\pi})$ for its class. When the genus ℓ is different from zero $\widehat{\pi}$ is non-trivial and the canonical map from $\mathrm{H}_2(\pi, \{\pi_j\})$ to $\mathrm{H}_2(\widehat{\pi})$ is an isomorphism identifying the fundamental classes. Write

$$E \colon F^2 \times \mathrm{Hom}(F, G) \to G^2$$

for the evaluation map, and let

$$\omega_c = \langle c, E^* \Omega \rangle,$$

the result of pairing c with the induced form, cf. [15 (13)]. This is a G-invariant 2-form on $\mathrm{Hom}(F, G)$. In view of [15 (15)] we have

$$d\omega_c = \langle \partial c, E^* \lambda \rangle. \tag{5.3}$$

We now apply a variant of the construction in Theorem 1 of [15]:

We pick c in such a way that

$$\partial c = [r] - [z_1] - \cdots - [z_n] \tag{5.4}$$

in the chain complex $C_*(F)$ of the nonhomogeneous reduced normalized bar resolution of F. This can always be done, cf. what is said about the chain (2.11) at the end of Section 2 above. In fact, for $\ell \geq 1$, the construction in [15], applied to the presentation

$$\langle x_1, y_1, \dots, x_\ell, y_\ell; r^\flat \rangle, \quad r^\flat = [x_1, y_1] \dots [x_\ell, y_\ell] = r z_1^{-1} \dots z_n^{-1}$$

of the fundamental group $\widehat{\pi}$ of $\widehat{\Sigma}$, yields a 2-chain c_1 with

$$\partial c_1 = [r z_1^{-1} \dots z_n^{-1}] \in C_1(F)$$

Since

$$\begin{aligned}
\partial[r\, z_n^{-1} \dots z_2^{-1} | z_1^{-1}] &= [z_1^{-1}] - [r\, z_n^{-1} \dots z_1^{-1}] + [r\, z_n^{-1} \dots z_2^{-1}] \\
\partial[r\, z_n^{-1} \dots z_3^{-1} | z_2^{-1}] &= [z_2^{-1}] - [r\, z_n^{-1} \dots z_2^{-1}] + [r\, z_n^{-1} \dots z_3^{-1}] \\
&\dots \\
\partial[r | z_n^{-1}] &= [z_n^{-1}] - [r\, z_n^{-1}] + [r],
\end{aligned}$$

adding to c_1 the 2-chains coming into play on the left-hand sides of these equations, we arrive at a 2-chain c_2 with

$$\partial c_2 = [r] + [z_1^{-1}] + \dots + [z_n^{-1}] \in C_1(F).$$

Finally, subtracting the 2-chains $[z_1 | z_1^{-1}], \dots, [z_n | z_n^{-1}]$ we obtain the desired 2-chain c satisfying (5.4) as asserted. When $\ell = 0$, c may be taken to be the negative of the sum of the chains

$$[z_1 \cdot \ldots \cdot z_{n-1} | z_n], \ [z_1 \cdot \ldots \cdot z_{n-2} | z_{n-1}], \ \dots \ , \ [z_1 | z_2].$$

Henceforth we suppose that the homotopy operator h on the forms on $\mathfrak{g}$ used to construct the various forms in [15] is the standard operator. Then the map ψ from $\mathfrak{g}$ to $\mathfrak{g}^*$, cf. Lemma 1 in [15], boils down to the adjoint of the symmetric bilinear form $\cdot$ on $\mathfrak{g}$. Let $\beta = h(\exp^*(\lambda))$, and define the 2-form $\omega_{c,\mathcal{P}}$ on $\mathcal{H}(\mathcal{P}, G)_{\mathbf{C}}$ by

$$\omega_{c,\mathcal{P}} = \eta^* \omega_c - \widehat{r}^* \beta. \tag{5.5}$$

Since $d\beta = \exp^*(\lambda)$, in view of (5.3) above,

$$d\omega_{c,\mathcal{P}} = -\overline{z}_1^* \lambda - \dots - \overline{z}_n^* \lambda \tag{5.6}$$

where we do not distinguish in notation between λ and its restrictions to the conjugacy classes $C_1, \dots, C_n$. Next, let

$$\mu = \psi \circ \widehat{r} \colon \mathcal{H}(\mathcal{P}, G)_{\mathbf{C}} \to \mathfrak{g}^*$$

that is, μ is the composite of the map $\widehat{r}$ from $\mathcal{H}(\mathcal{P}, G)_{\mathbf{C}}$ to $\mathfrak{g}$ with the adjoint ψ of the symmetric bilinear form $\cdot$ from $\mathfrak{g}$ to its dual; here we do not distinguish in notation between a map into O and its composite with the inclusion into $\mathfrak{g}$. Recall

$$\delta_G \lambda = -d\vartheta \quad [15\ (6)].$$

As in [15], we write

$$\mu^\sharp\colon \mathfrak g \to C^\infty(\mathcal H(\mathcal P, G)_{\mathbf C})$$

for the adjoint of μ. We now assert

$$\delta_G \omega_{c,\mathcal P} = d\mu^\sharp - \overline z_1^* \vartheta - \dots - \overline z_n^* \vartheta \tag{5.7}$$

where we do not distinguish in notation between ϑ and its restrictions to the conjugacy classes $C_1, \dots, C_n$. Indeed, cf. the proof of Theorem 2 of [15],

$$\psi^\sharp = h \circ (\exp^* \vartheta - \delta_G(\beta))$$

and

$$\delta_G(\Omega) = \delta\vartheta \quad [15\ (4)]$$

whence

$$\begin{aligned}
\delta_G \omega_{c,\mathcal P} &= \delta_G(\eta^* \omega_c - \widehat r^* \beta)\\
&= \eta^* \delta_G \omega_c - \widehat r^* \delta_G \beta\\
&= \eta^* \langle \partial c, E^* \vartheta\rangle - \widehat r^* \delta_G \beta\\
&= \widehat r^* (\exp^* \vartheta - \delta_G \beta) - \overline z_1^* \vartheta - \dots - \overline z_n^* \vartheta\\
&= d\mu^\sharp - \overline z_1^* \vartheta - \dots - \overline z_n^* \vartheta
\end{aligned}$$

as asserted. The formulas (5.6) and (5.7) show that μ is somewhat like a momentum mapping for the (non-closed) 2-form $\omega_{c,\mathcal P}$, up to certain error terms. In Section 7 below we shall add appropriate forms which will correct this error. Before we can do so we need some preparation to which the next Section is devoted.

6. A single conjugacy class

Let C be a conjugacy class of G and $\mathcal O$ an adjoint orbit which is mapped onto C under the exponential mapping from $\mathfrak g$ to G. Let $X, Y \in \mathfrak g$. The vector fields $X_{\mathcal O}$ and $Y_{\mathcal O}$ on $\mathcal O$ generated by X and Y are given by the assignment to a point $Z \in \mathcal O$ of $[X, Z] \in \mathrm T_Z \mathcal O$ and $[Y, Z] \in \mathrm T_Z \mathcal O$, and the "Kirillov" form ω on $\mathcal O$ is given by the expression

$$\omega_Z(X_{\mathcal O}, Y_{\mathcal O}) = \omega_Z([X, Z], [Y, Z]) = [X, Y] \cdot Z = [Z, X] \cdot Y. \tag{6.1}$$

Notice that there is no need to assume the symmetric bilinear form $\cdot$ on $\mathfrak g$ to be nondegenerate; just take the 2-form ω on $\mathcal O$ defined by (6.1). For a point p of C, an arbitrary tangent vector is of the form

$$Xp - pX = (X - \mathrm{Ad}(p)X)p \in \mathrm T_p C,$$

where $\cdot p$ and $p\cdot$ denote the effect of right and left translation and where X is an element of the Lie algebra $\mathfrak g$, identified with the tangent space $\mathrm T_e G$ of G at e. As before, let $\beta = h(\exp^* \lambda)$ where λ denotes Cartan's fundamental 3-form on G.

Theorem 6.2. *The assignment*

$$\tau(Xp - pX, Yp - pY) = \frac{1}{2}(X \cdot \mathrm{Ad}(p)Y - Y \cdot \mathrm{Ad}(p)X), \quad p \in C, \tag{6.2.1}$$

yields an equivariant 2-form τ *on* C *having the property*

$$\exp^* \tau = \beta - \omega. \tag{6.2.2}$$

Before proving the theorem, we spell out the following which will be crucial:

Corollary 6.3. *The 2-form* τ *satisfies the formulas*

$$d\tau = \lambda \tag{6.3.1}$$

$$\delta_G \tau = \vartheta\,. \tag{6.3.2}$$

Proof of the Corollary. Since ω is closed,

$$\exp^*(d\tau) = d\exp^* \tau = d(\beta - \omega) = d\beta = \exp^* \lambda$$

whence $d\tau = \lambda$. Furthermore, denote by J the composite of the inclusion of $\mathcal{O}$ into $\mathfrak{g}$ with the adjoint of the given symmetric bilinear form; formula (6.1) says that

$$\omega_Z(X_{\mathcal{O}}, Y_{\mathcal{O}}) = d(X \circ J)_Z(Y_{\mathcal{O}}),$$

that is, with our definition of the operator δ_G involving the negative (!) of the contraction operator, cf. Section 1 of [15], we have

$$\delta_G \omega = -J^\sharp.$$

On the other hand, in view of the formula

$$\delta_G \lambda = -d\vartheta \quad [15\ (4)],$$

on the whole Lie algebra $\mathfrak{g}$, we get

$$\begin{aligned}
\delta_G(\beta) &= \delta_G(h\exp^* \lambda) \\
&= -h(\delta_G(\exp^* \lambda)) \\
&= -h(\exp^* \delta_G(\lambda)) \\
&= h(\exp^* d(\vartheta)) \\
&= hd(\exp^*(\vartheta)) \\
&= \exp^*(\vartheta) - dh(\exp^*(\vartheta)) \\
&= \exp^*(\vartheta) - \psi^\sharp.
\end{aligned}$$

Consequently, on $\mathcal{O}$, where ψ amounts to J, we obtain

$$\exp^*(\delta_G \tau) = \delta_G(\exp^* \tau) = \delta_G(\beta - \omega) = \exp^*(\vartheta) - \psi^\sharp + J^\sharp = \exp^*(\vartheta)$$

whence

$$\delta_G \tau = \vartheta$$

holds on C as asserted. □

REMARK. Notice that the proof of (6.3) works whether or not the restriction of the exponential mapping to $\mathcal{O}$ is a diffeomorphism.

We now begin with the preparations for the proof of Theorem 6.2. Recall [14 II.1.7] that the derivative at $Z \in \mathfrak{g}$ of the exponential mapping exp from $\mathfrak{g}$ to G is given by the formula

$$\mathrm{dexp}_Z = d(L_{\exp Z})_e \circ \frac{1 - e^{-\mathrm{adZ}}}{\mathrm{ad}Z} \tag{6.4.1}$$

that is,

$$\mathrm{dexp}_Z = d(L_{\exp Z})_e \circ \left(1 - \frac{1}{2}\mathrm{ad}Z + \frac{1}{3!}(\mathrm{ad}Z)^2 - \frac{1}{4!}(\mathrm{ad}Z)^3 + \dots\right). \tag{6.4.2}$$

We now consider the exponential mapping exp from $\mathcal{O}$ to the corresponding conjugacy class C. Let $p = \exp(Z) \in C \subseteq G$. The above formula entails that the derivative

$$\mathrm{dexp}_Z \colon \mathrm{T}_Z\mathcal{O} \longrightarrow \mathrm{T}_p C$$

of the exponential mapping sends the tangent vector $[X, Z]$ to the tangent vector

$$Xp - pX = (X - \mathrm{Ad}(p)X)p \in \mathrm{T}_p C.$$

Consequently the statement of Theorem 6.2 is equivalent to the following.

Lemma 6.5. *The 2-form β on $\mathcal{O}$ is given by the formula*

$$\beta_Z([X,Z],[Y,Z]) = [Z,X]\cdot Y + \frac{1}{2}(X\cdot \mathrm{Ad}(p)Y - Y\cdot \mathrm{Ad}(p)X).$$

Henceforth we write $[\cdot,\cdot,\cdot]$ for the triple product.

Proof. As in [15], write $\rho = \exp^*\lambda$, where λ refers to the fundamental 3-form on G. For simplicity, write $p_t = \exp(tZ) \in G$. We then have

$$\begin{aligned}
2\beta_Z([X,Z],[Y,Z]) &= 2\int_0^1 \rho_{tZ}(Z,[X,tZ],[Y,tZ])dt \\
&= \int_0^1 \left[p_t^{-1}(\mathrm{dexp}_{tZ}(Z)), \mathrm{Ad}(p_t^{-1})X - X, \mathrm{Ad}(p_t^{-1})Y - Y\right] dt \\
&= \int_0^1 \left[Z, \mathrm{Ad}(p_t^{-1})X - X, \mathrm{Ad}(p_t^{-1})Y - Y\right] dt \\
&= \int_0^1 [Z,X,Y]\,dt + \int_0^1 \left[Z, \mathrm{Ad}(p_t^{-1})X, \mathrm{Ad}(p_t^{-1})Y\right] dt \\
&\quad + \int_0^1 [Z, \mathrm{Ad}(p_t^{-1})X, -Y]dt + \int_0^1 [Z, -X, \mathrm{Ad}(p_t^{-1})Y]dt \\
&= 2[Z,X]\cdot Y \\
&\quad + \int_0^1 (\mathrm{Ad}(\exp(tZ)X, Z]\cdot Y - [\mathrm{Ad}(\exp(tZ)Y, Z]\cdot X)\,dt.
\end{aligned}$$

However,

$$[\mathrm{Ad}(\exp(tZ)Y, Z] = -\left([Z,Y] + t[Z,[Z,Y]] + \frac{t^2}{2!}[Z,[Z,[Z,Y]]] + \dots\right)$$

whence

$$\begin{aligned}\int_0^1 [\mathrm{Ad}(\exp(tZ)Y, Z]dt &= -\left([Z,Y] + \frac{1}{2}[Z,[Z,Y]] + \frac{1}{3!}[Z,[Z,[Z,Y]]] + \dots\right)\\ &= Y - e^{\mathrm{ad}(Z)}Y\\ &= Y - \mathrm{Ad}(p)Y\end{aligned}$$

and likewise

$$\int_0^1 [\mathrm{Ad}(\exp(tZ)X, Z]dt = X - \mathrm{Ad}(p)X$$

whence the assertion. □

7. The completion of the construction

In view of (5.6), (5.7), and (6.3) above, very little work remains to prove the following.

Theorem 7.1. *The equivariant 2-form*

$$\omega_{c,\mathcal{P},\mathbf{C}} = \omega_{c,\mathcal{P}} + \overline{z}_1^*\tau + \dots + \overline{z}_n^*\tau \tag{7.1.1}$$

on $\mathcal{H}(\mathcal{P},G)_{\mathbf{C}}$ is closed, and the adjoint $\mu^\sharp$ from $\mathfrak{g}$ to $C^\infty(\mathcal{H}(\mathcal{P},G)_{\mathbf{C}})$ of the smooth equivariant map μ from $\mathcal{H}(\mathcal{P},G)_{\mathbf{C}}$ to $\mathfrak{g}^*$ satisfies the identity

$$\delta_G\omega_{c,\mathcal{P},\mathbf{C}} = d\mu^\sharp. \tag{7.1.2}$$

Consequently the difference $\omega_{c,\mathcal{P},\mathbf{C}} - \mu^\sharp$ is an equivariantly closed form in $(\Omega_G^{*,*}(\mathcal{H}(\mathcal{P},G)_{\mathbf{C}}); d, \delta_G)$ of total degree 2.

The identity (7.1.2) says that, for every $X \in \mathfrak{g}$,

$$-\omega_{c,\mathcal{P},\mathbf{C}}(X_{\mathcal{H}}, \cdot\,) = d(X \circ \mu),$$

that is, μ is formally a momentum mapping for the G-action on $\mathcal{H}(\mathcal{P},G)_{\mathbf{C}}$, with reference to $\omega_{c,\mathcal{P},\mathbf{C}}$, except that the latter is not necessarily nondegenerate; here we have written $X_{\mathcal{H}}$ for the vector field on $\mathcal{H}(\mathcal{P},G)_{\mathbf{C}}$ induced by $X \in \mathfrak{g}$.

8. Groupoids and the equivariantly closed form

Let $\phi \in \mathrm{Hom}(F,G)_{\mathbf{C}}$ and suppose that $\phi(r)$ lies in the centre of G. The above construction (3.6), applied to the present data, yields an alternating bilinear form

$$\omega_{\kappa,\cdot,\phi}\colon \mathrm{H}^1_{\mathrm{par}}(\pi,\{\pi_j\};\mathfrak{g}_\phi)\otimes \mathrm{H}^1_{\mathrm{par}}(\pi,\{\pi_j\};\mathfrak{g}_\phi)\longrightarrow \mathbf{R} \tag{8.1}$$

on $\mathrm{H}^1_{\mathrm{par}}(\pi,\{\pi_j\};\mathfrak{g}_\phi)$ which is symplectic, i. e. nondegenerate, provided that $\cdot$ is nondegenerate. Next, let

$$\omega_{c,\mathbf{C}} = \omega_c + \overline{z}_1^*\tau + \cdots + \overline{z}_n^*\tau. \tag{8.2}$$

This is a 2-form on $\mathrm{Hom}(F,G)_{\mathbf{C}}$ whose restriction to $\mathrm{Hom}(\pi,G)_{\mathbf{C}}$ coincides with the restriction of $\omega_{c,\mathcal{P},\mathbf{C}}$ to $\mathrm{Hom}(\pi,G)_{\mathbf{C}}$ where $\mathrm{Hom}(\pi,G)_{\mathbf{C}}$ is viewed as a subspace of $\mathcal{H}(\mathcal{P},G)_{\mathbf{C}}$ as explained above. Henceforth we denote by K_ϕ the kernel of the derivative $\mathrm{T}_\phi r$ occurring in (4.5.1) above. Our present goal is to prove the following.

Theorem 8.3. *Right translation identifies the restriction of the 2-form $\omega_{c,\mathbf{C}}$ to K_ϕ with the alternating bilinear form on $Z^1_{\mathrm{par}}(\pi,\{\pi_j\};\mathfrak{g}_\phi)$ obtained as the composite of $\omega_{\kappa,\cdot,\phi}$ with the projection from $Z^1_{\mathrm{par}}(\pi,\{\pi_j\};\mathfrak{g}_\phi)$ to $\mathrm{H}^1_{\mathrm{par}}(\pi,\{\pi_j\};\mathfrak{g}_\phi)$.*

Key Lemma 8.4. *For an arbitrary real representation V of π with an invariant symmetric bilinear form $\cdot$, the value of the alternating bilinear form* (3.6) *on $\mathrm{H}^1_{\mathrm{par}}(\pi,\{\pi_j\};V)$ for two parabolic V-valued 1-cocycles u and v with*

$$u(z_j) = z_jX_j - X_j \quad \text{and} \quad v(z_j) = z_jY_j - Y_j, \qquad X_j, Y_j \in V,\ 1\le j\le n, \tag{8.4.1}$$

is given by the formula

$$\omega_V([u],[v]) = \langle c, u\cup v\rangle + \frac{1}{2}\sum\left(X_j\cdot z_jY_j - Y_j\cdot z_jX_j\right). \tag{8.4.2}$$

We postpone the proof for the moment and now give the

Proof of Theorem 8.3. The chosen 2-chain $c\in C_2(F)$ in the nonhomogeneous reduced normalized bar resolution for F looks like

$$c = \sum \nu_{j,k}[x_j|x_k].$$

Define the bilinear form $\omega_{c,\cup,\phi}$ on

$$Z^1(F,\mathfrak{g}_\phi) = C^1(\mathcal{P},\mathfrak{g}_\phi) \quad (= C^1(\widehat{\mathcal{P}},\mathfrak{g}_\phi))$$

(cf. (5.1)) by the explicit formula

$$\omega_{c,\cup,\phi}(u,v) = \langle c, u\cup v\rangle = \sum \nu_{j,k}u(x_j)\cdot(\mathrm{Ad}(\phi(x_j))v(x_k)), \quad u,v\in Z^1(F,\mathfrak{g}_\phi).$$

By [15 (4.6)], the 2-form ω_c is the right translation of the antisymmetrization of $\omega_{c,\cup,\phi}$ and hence the two coincide on K_ϕ. This involves the Alexander-Whitney diagonal map [28] (VIII.9 Ex. 1, p. 248). Inspection shows that, still on K_ϕ, the

remaining terms $\overline{z}_1^*\tau, \cdots, \overline{z}_n^*\tau$ in (8.2) correspond precisely to the remaining terms in (8.4.2), with $V = \mathfrak{g}_\phi$. In view of Lemma 8.4, this proves Theorem 8.3. $\square$

The proof of the Key Lemma relies on a detailed analysis of the multiplicative structure of the cohomology of the group system in terms of the fundamental groupoid $\widetilde{\pi} = \Pi(\Sigma; p_0, p_1, \ldots, p_n)$ introduced in Section 2 above. We now explain this.

Let $\widetilde{F}$ denote the groupoid which is free on the generators of (2.4). To have a neutral notation, whenever necessary, we shall write $\widetilde{\Pi}$ for either $\widetilde{F}$ or $\widetilde{\pi}$; accordingly we write Π for either F or π. As usual, view G as a groupoid with a single object which we write e. Write $\mathrm{Hom}(\widetilde{\Pi}, G)$ for the space of groupoid homomorphisms from $\widetilde{\Pi}$ to G.

The assignments

$$
\begin{gathered}
i(e) = p_0, \quad i(x_j) = x_j, \quad i(y_j) = y_j, \quad i(z_j) = \gamma_j a_j \gamma_j^{-1}, \\
\mathrm{ret}(p_j) = e, \quad \mathrm{ret}(x_j) = x_j, \quad \mathrm{ret}(y_j) = y_j, \quad \mathrm{ret}(a_j) = z_j, \quad \mathrm{ret}(\gamma_j) = \mathrm{Id},
\end{gathered}
$$

yield obvious functors $i\colon \Pi \to \widetilde{\Pi}$ and $\mathrm{ret}\colon \widetilde{\Pi} \to \Pi$ inducing a deformation retraction of $\widetilde{\Pi}$ onto Π; cf. e. g. [8] (6.5.13) for this notion. These functors induce maps

$$
i^*\colon \mathrm{Hom}(\widetilde{\Pi}, G) \longrightarrow \mathrm{Hom}(\Pi, G), \quad \mathrm{ret}^*\colon \mathrm{Hom}(\Pi, G) \longrightarrow \mathrm{Hom}(\widetilde{\Pi}, G)
$$

which, for $\Pi = F$, are manifestly smooth. We shall occasionally refer to i^* and ret^* as *restriction* and *corestriction*, respectively.

The obvious action of G on $\mathrm{Hom}(\Pi, G)$ by conjugation extends to an action of the group $G^{\widetilde{\Pi}_0} \cong G \times \cdots \times G$ ($n+1$ copies of G) on $\mathrm{Hom}(\widetilde{\Pi}, G)$ in the following way: We denote by s and t the source and target mappings from $\widetilde{\Pi}$ to $\widetilde{\Pi}_0$. Given a homomorphism α from $\widetilde{\Pi}$ to G and $\vartheta \in G^{\widetilde{\Pi}_0}$, the homomorphism $\vartheta\alpha$ is defined by

$$
\vartheta\alpha(w) = \vartheta(t(w))\alpha(w)(\vartheta(s(w)))^{-1}.
$$

The orbit space for the $G^{\widetilde{\Pi}_0}$-action on $\mathrm{Hom}(\widetilde{\Pi}, G)$ will be denoted by $\mathrm{Rep}(\widetilde{\Pi}, G)$.

As in the group case, we denote by $\mathrm{Hom}(\widetilde{\Pi}, G)_{\mathbf{C}}$ the space of homomorphisms χ from $\widetilde{\Pi}$ to G for which the value $\chi(a_j)$ of each generator a_j lies in C_j, for $1 \leq j \leq n$. The $G^{\widetilde{\Pi}_0}$-action on $\mathrm{Hom}(\widetilde{\Pi}, G)$ leaves the subspace $\mathrm{Hom}(\widetilde{\Pi}, G)_{\mathbf{C}}$ invariant, and we denote by $\mathrm{Rep}(\widetilde{\Pi}, G)_{\mathbf{C}}$ the orbit space for this action. Since

$$
\chi(z_j) = \chi(\gamma_j a_j \gamma_j^{-1}) = \chi(\gamma_j)\chi(a_j)\chi(\gamma_j)^{-1}, \quad j = 1, \ldots, n,
$$

the condition '$\chi(a_j) \in C_j$' is equivalent to the condition '$\chi(z_j) \in C_j$', which we used in the group case.

Proposition 8.5. *The restriction mapping induces bijections*

$$
i^*\colon \mathrm{Rep}(\widetilde{\Pi}, G) \longrightarrow \mathrm{Rep}(\Pi, G), \quad i^*\colon \mathrm{Rep}(\widetilde{\Pi}, G)_{\mathbf{C}} \longrightarrow \mathrm{Rep}(\Pi, G)_{\mathbf{C}}.
$$

Thus we can study the structure of $\mathrm{Rep}(\Pi, G)_{\mathbf{C}}$ by looking at $\mathrm{Rep}(\widetilde{\Pi}, G)_{\mathbf{C}}$ instead. In particular, this remark applies to the infinitesimal structure, in the following

way: Write β for the nonhomogeneous unreduced normalized bar resolution. The retraction ret from $\widetilde{\Pi}$ to Π induces a deformation retraction from the nerve $N\widetilde{\Pi}$ of $\widetilde{\Pi}$ to the nerve $N\Pi$ of Π, and a canonical section for the latter is of course induced by the injection i from Π to $\widetilde{\Pi}$; hence ret induces a deformation retraction from $\beta\widetilde{\Pi}$ to $\beta\Pi$. In particular, $\beta\widetilde{\Pi}$ yields a free resolution of R in the category of left $R\Pi$-modules of the kind written A in Section 1 above. In fact, write $\widetilde{\pi}_\partial$ for the free subgroupoid of $\widetilde{F}$ having $p_1,\dots,p_n$ as objects and $a_1,\dots,a_n$ as free generators for its morphisms; this groupoid may also be viewed as a subgroupoid of $\widetilde{\pi}$, and we do not distinguish in notation between the two subgroupoids. Abstractly, $\widetilde{\pi}_\partial$ amounts of course to a disjoint union of the n free cyclic groups $\pi_1,\dots,\pi_n$, and

$$\beta\widetilde{\pi}_\partial = \oplus_{j=1}^n \beta\pi_j \subseteq \beta\widetilde{\pi}$$

in such a way that extension of scalars yields an injection of $B = \oplus_{j=1}^n R\pi \otimes_{\pi_j} \beta\pi_j$ of $R\pi$-complexes onto a direct summand of $\beta\widetilde{\pi}$. The $R\pi$-complex B plays exactly the same role as that denoted in Section 1 above by the same symbol, we have the split exact sequence (1.1) at our disposal, and the quotient

$$\beta(\widetilde{\pi},\widetilde{\pi}_\partial) = A\big/B = \beta\widetilde{\pi}\big/B$$

computes the relative cohomology $\mathrm{H}^*(\widetilde{\pi},\widetilde{\pi}_\partial;\cdot)$. With the present interpretation of (A,B) as resolution over the group system $(\pi;\pi_1,\dots,\pi_n)$, the relative cohomology $\mathrm{H}^*(\widetilde{\pi},\widetilde{\pi}_\partial;\cdot)$ *coincides* with the cohomology $\mathrm{H}^*(\pi,\{\pi_j\};\cdot)$ of the group system $(\pi;\pi_1,\dots,\pi_n)$, though. In particular, the standard formula for the diagonal in the nonhomogeneous unreduced normalized bar resolution $\beta\widetilde{\pi}$ for $\widetilde{\pi}$ yields the multiplicative structure of $\mathrm{H}^*(\pi,\{\pi_j\};\cdot)$.

Proof of the Key Lemma 8.4. Let u and v be parabolic V-valued 1-cocycles on $\beta\pi$ so that (8.4.1) holds; the calculation of the value $\omega_V([u],[v])$ of the pairing (3.6) and hence (8.1) may now be split into the following steps:

(1) **Extension:** The composites $u' = u\circ \mathrm{ret}$ and $v' = v\circ\mathrm{ret}$ of u and v, respectively, with the retraction ret from $\beta\widetilde{\pi}$ to $\beta\pi$ yields extensions to parabolic V-valued 1-cocycles u' and v' on $\beta\widetilde{\pi}$.
(2) **Normalization:** Normalize u' and v' to obtain groupoid cocycles $\widetilde{u}$ and $\widetilde{v}$ which take the value zero on the peripheral part B of the resolution, cf. Section 1 above; notice that this amounts to the requirement that $\widetilde{u}$ and $\widetilde{v}$ vanish on the n boundary circles $S_1,\dots,S_n$.
(3) **Lifting:** Lift the 2-chain $c\in C_2(F)$ of the nonhomogeneous reduced normalized bar resolution for F to a 2-chain $\widetilde{c}\in C_2(\widetilde{F})$ of the nonhomogeneous reduced normalized bar resolution for $\widetilde{F}$ which (i) passes to a relative cycle for $(\widetilde{\pi},\widetilde{\pi}_\partial)$ and which (ii) under the retraction ret from $C_*(\widetilde{F})$ to $C_*(F)$ goes to c.
(4) **Computation:** The value $\omega_V([u],[v])$ is then computed by the formula

$$\omega_V([u],[v]) = \langle \widetilde{c}, \widetilde{u}\cup\widetilde{v}\rangle. \tag{8.6}$$

We note that the groupoid description is crucial for the normalization in step 2; such a normalization would be impossible for the group cocycles u and v, and a calculation of the value $\omega_V([u],[v])$ directly in terms of u and v would lead to a mess. Moreover, by a purely formal reasoning in the relative cohomology of the pair (A,B) or, what amounts to the same, of the pair $(\Sigma, \partial\Sigma)$, the right-hand side of (8.6) is well defined and yields the pairing (3.6).

There is no more need to comment on step 1, and we now explain the other steps.

Step 2. With reference to (8.4.1), let X and Y be the groupoid 0-cocycles defined by

$$X(p_0)=0, \quad X(p_j)=X_j, \quad Y(p_0)=0, \quad Y(p_j)=Y_j, \quad 1 \le j \le n.$$

Define $\widetilde{u}$ and $\widetilde{v}$ by

$$\widetilde{u} = u' - \delta X = u \circ \mathrm{ret} - \delta X, \quad \widetilde{v} = v' - \delta Y = v \circ \mathrm{ret} - \delta Y.$$

Step 3. View c as a 2-chain of $\widetilde{F}$ by the embedding of F into $\widetilde{F}$ and let

$$\widetilde{c} = c + \sum_j \left([\gamma_j^{-1}|\gamma_j a_j] - [\gamma_j a_j|\gamma_j^{-1}]\right). \tag{8.7}$$

Then

$$\partial \widetilde{c} = [\widetilde{r}] - [a_1] - \cdots - [a_n]$$

whence, in particular, $\widetilde{c}$ is manifestly a relative 2-cycle for $(\widetilde{\pi}, \widetilde{\pi}_\partial)$. Notice that c itself is *not* a relative cycle for $(\widetilde{\pi}, \widetilde{\pi}_\partial)$.

Step 4. By definition

$$\omega_V([u],[v]) = \langle \widetilde{c}, \widetilde{u} \cup \widetilde{v} \rangle =$$

$$\langle c, (u' - \delta X) \cup (v' - \delta Y) \rangle \tag{8.8}$$

$$+ \sum_j \langle [\gamma_j^{-1}|\gamma_j a_j], (u' - \delta X) \cup (v' - \delta Y) \rangle \tag{8.9}$$

$$- \sum_j \langle [\gamma_j a_j|\gamma_j^{-1}], (u' - \delta X) \cup (v' - \delta Y) \rangle \tag{8.10}$$

The term (8.8) is the sum

$$\langle c, u' \cup v' \rangle - \langle c, u' \cup \delta Y \rangle - \langle c, \delta X \cup v' \rangle + \langle c, \delta X \cup \delta Y \rangle \tag{8.11}$$

Since c is a group chain, $\langle c, u' \cup v' \rangle$ equals $\langle c, u \cup v \rangle$. Further, since u' is a cocycle,

$$\begin{aligned}\langle c, u' \cup \delta Y \rangle &= \langle c, -\delta(u' \cup Y) \rangle = \langle -\partial c, u' \cup Y \rangle \\ &= \sum_j \langle [z_j], u' \cup Y \rangle = \sum_j u(z_j) \cdot Y(p_0) = 0\end{aligned}$$

since $Y(p_0) = 0$. Likewise, the last two terms in (8.11) involve $X(p_0)$, which is zero as well, so we are left with $\langle c, u \cup v\rangle$ as the value of the entire sum (8.11) which computes (8.8). The term (8.9) involves n factors which may be computed as

$$\langle [\gamma_j^{-1}|\gamma_j a_j], (u' - \delta X) \cup (v' - \delta Y)\rangle = (u' - \delta X)(\gamma_j^{-1}) \cdot (\gamma_j^{-1}((v' - \delta Y)(\gamma_j a_j))).$$

However, $u'(\gamma_j^{-1}) = u(\mathrm{ret}\gamma_j^{-1}) = u(e)$ which is zero,

$$\begin{aligned} \delta X(\gamma_j^{-1}) &= X(\partial\gamma_j^{-1}) = X(p_0 - p_j) = -X_j, \\ v'(\gamma_j a_j) &= v(\mathrm{ret}(\gamma_j a_j)) = v(z_j) = z_j Y_j - Y_j, \\ \delta Y(\gamma_j a_j) &= Y(\partial(\gamma_j a_j)) = Y(a_j p_j - p_0) = z_j Y_j, \end{aligned}$$

and γ_j and γ_j^{-1} act as the identity on V. Hence (8.9) equals

$$\sum_j X_j \cdot (z_j Y_j - Y_j - z_j Y_j) = -\sum_j X_j \cdot Y_j.$$

Finally the term (8.10) involves n factors which may be computed as

$$\begin{aligned} \langle [\gamma_j a_j|\gamma_j^{-1}], &(u' - \delta X) \cup (v' - \delta Y)\rangle \\ &= ((u' - \delta X)(\gamma_j a_j)) \cdot (\gamma_j a_j)\left((v' - \delta Y)(\gamma_j^{-1})\right) \\ &= (u(\mathrm{ret}(\gamma_j a_j)) - \delta X(\gamma_j a_j)) \cdot (\gamma_j a_j)\left(v(\mathrm{ret}(\gamma_j^{-1})) - \delta Y(\gamma_j^{-1})\right) \\ &= (u(z_j) - X(a_j p_j - p_0)) \cdot (\gamma_j a_j)\left(-Y(\partial\gamma_j^{-1})\right) \\ &= (z_j X_j - X_j - z_j X_j) \cdot z_j Y_j \\ &= -X_j \cdot z_j Y_j \end{aligned}$$

Consequently

$$\omega_V([u],[v]) = \langle c, u \cup v\rangle + \sum X_j \cdot (z_j Y_j - Y_j).$$

Likewise,

$$\omega_V([v],[u]) = \langle c, v \cup u\rangle + \sum Y_j \cdot (z_j X_j - X_j).$$

By antisymmetry,

$$\begin{aligned} 2\omega_V([u],[v]) &= \omega_V([u],[v]) - \omega_V([v],[u]) \\ &= 2\langle c, u \cup v\rangle + \sum X_j \cdot (z_j Y_j - Y_j) - \sum Y_j \cdot (z_j X_j - X_j) \\ &= 2\langle c, u \cup v\rangle + \sum (X_j \cdot z_j Y_j - Y_j \cdot z_j X_j) \end{aligned}$$

This proves the key Lemma. □

9. Stratified symplectic structures

Suppose that is G compact and that the symmetric bilinear form $\cdot$ on $\mathfrak{g}$ is nondegenerate. Recall that the notion of a stratified symplectic space has been introduced in [31].

Theorem 9.1. *With respect to the decomposition according to G-orbit types, the space* $\mathrm{Rep}(\pi, G)_{\mathbf{C}}$ *inherits a structure of a stratified symplectic space.*

In fact, the argument for the main result of [31] shows that each connected component of a reduced space of the kind considered inherits a structure of a stratified symplectic space. In the setting of [31] the hypothesis of properness is used *only* to guarantee that the reduced space is in fact connected. In our situation, we know a priori that the reduced space is connected.

Corollary 9.2. *The space* $\mathrm{Rep}(\pi, G)_{\mathbf{C}}$ *has a unique open, connected, and dense stratum.*

In fact, this follows at once from [31] (5.9). The stratum mentioned in the corollary is called the *top stratum*. Thus there is a certain subgroup T of G, unique up to conjugacy, such that every $\phi \in \mathrm{Hom}(\pi, G)_{\mathbf{C}}$ representing a point of the top stratum has stabilizer Z_ϕ conjugate to T. In many cases, T is just the centre of G, and the top stratum consists of representations which are *irreducible* in the sense that the stabilizer has Lie algebra the Lie algebra of the centre of G (but the top stratum may be smaller than the space of irreducible representations). See [18] for details. There may be *no* irreducible representations at all, though. This happens for example when π is abelian and G non-abelian.

10. Relationship with gauge theory constructions

For G compact, and closed [2,12] as well as punctured [4,5] surfaces, the symplectic structure on the top stratum of the corresponding space of gauge equivalence classes of flat connections has been described using methods of gauge theory. In this section we show that the usual identification of this space with the corresponding space of representations identifies the symplectic structures on the top strata. This extends what is done in Section 6 of [19] where, for the case of a closed surface, the symplectic structures on all strata have been shown to correspond to each other.

Let G be a general, not necessarily compact Lie group. The compactness hypothesis will be made at the appropriate stage. Our surface Σ is compact, with $n \geq 0$ boundary circles $S_1, \dots, S_n$ and chosen base point p_0. Write $\Sigma^\bullet$ for the corresponding *punctured* surface with base point p_0 which contains Σ as a based deformation retract in such a way that each S_j is a circle about the corresponding puncture. We do not exclude the case of a closed surface and we agree that in this case $\Sigma^\bullet$ and Σ coincide.

Let $\xi \colon P \to \Sigma^\bullet$ be a flat principal G-bundle, having the structure group G act from the right as usual, and pick a base point $\widehat{p}_0$ of P with $\xi(\widehat{p}_0) = p_0$. In many cases, for example when π is a free group or when G is simply connected, ξ will be topologically trivial. Write $\mathcal{A}(\xi)$ for the space of connections on ξ. The assignment to a gauge transformation ν on ξ of $x_\nu \in G$ defined by $\nu(\widehat{p}_0) = \widehat{p}_0 x_\nu$ furnishes a homomorphism from the group $\mathcal{G}(\xi)$ of gauge transformations onto G. Among the

various descriptions of the space $\Omega^*(\Sigma^\bullet, \mathrm{ad}(\xi))$ of forms with values in the adjoint bundle

$$\mathrm{ad}(\xi)\colon P \times_G \mathfrak{g} \longrightarrow \Sigma^\bullet$$

we shall take that in terms of G-invariant horizontal $\mathfrak{g}$-valued forms on P; we note that here G acts on its Lie algebra $\mathfrak{g}$ from the *left* by the adjoint representation $\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathfrak{g})$ as usual.

For a smooth closed path $w\colon I \to \Sigma^\bullet$ defined on the unit interval I, with starting point $p_0 \in \Sigma^\bullet$, the *holonomy* $\mathrm{Hol}_{w,\widehat{p}_0}(A) \in G$ of A *with reference to* $\widehat{p}_0$ is defined by

$$\widehat{w}(1) = \widehat{p}_0 \, \mathrm{Hol}_{w,\widehat{p}_0}(A) \in P$$

where $\widehat{w}$ refers to the *horizontal* lift of w having starting point $\widehat{p}_0$. For $b \in G$, we denote by L_b the operation of left translation from $\mathfrak{g}$ to $\mathrm{T}_b G$.

We maintain the notation of Sections 2 and 4 above. In particular, F is the free group on the generators $x_1, y_1, \ldots, x_\ell, y_\ell, z_1, \ldots, z_n$ of (2.1) and, with an abuse of notation, the corresponding closed (edge) paths in Σ representing these generators are denoted by the same symbols. The assignment to a connection A of the point

$$(\mathrm{Hol}_{x_1,\widehat{p}_0}(A), \mathrm{Hol}_{y_1,\widehat{p}_0}(A), \ldots, \mathrm{Hol}_{x_\ell,\widehat{p}_0}(A), \mathrm{Hol}_{y_\ell,\widehat{p}_0}(A), \mathrm{Hol}_{z_1,\widehat{p}_0}(A), \ldots, \mathrm{Hol}_{z_n,\widehat{p}_0}(A))$$

of $G^{2\ell+n}$ yields a smooth map

$$\rho\colon \mathcal{A}(\xi) \longrightarrow G^{2\ell+n} = \mathrm{Hom}(F, G)$$

which is $\mathcal{G}(\xi)$-equivariant in the sense that

$$\rho(\nu A) = x_\nu \rho(A) x_\nu^{-1}, \qquad \text{for every gauge transformation} \quad \nu;$$

this map is referred to as *Wilson loop mapping* in [19], where a comment is made as to the appropriate interpretation of the property of ρ being smooth. The restriction of ρ to the subspace $\mathcal{F}(\xi)$ of flat connections yields the standard map from $\mathcal{F}(\xi)$ to $\mathrm{Hom}(\pi, G)$, viewed as a subspace of $\mathrm{Hom}(F, G)$ via the projection from F to π, and this map depends only on the choice of $\widehat{p}_0$ but *not* on the choices of closed paths representing the generators of F. The induced map from the space of gauge equivalence classes of flat connections on ξ to (the corresponding open and closed subset of) the representation space $\mathrm{Rep}(\pi, G) = \mathrm{Hom}(\pi, G)/G$ is then independent of the choice of $\widehat{p}_0$.

Now let A be a flat connection, and let $\phi = \rho(A)$ be the corresponding homomorphism from π to G. Write $\Omega^1 = \Omega^1(\Sigma^\bullet, \mathrm{ad}(\xi))$ and $\mathrm{T}_\phi = \mathrm{T}_\phi G^{2\ell+n}$. Consider the universal cover $\widetilde{\Sigma}^\bullet$ of $\Sigma^\bullet$, and suppose things arranged in such a way that π acts on $\widetilde{\Sigma}^\bullet$ and $\widetilde{\Sigma}$ from the *right*. As usual, this action is related to the corresponding action of π from the left by $x\widetilde{p} = \widetilde{p}x^{-1}$, for $x \in \pi$ and $\widetilde{p} \in \widetilde{\Sigma}^\bullet$. After a choice o of base point of $\widetilde{\Sigma}^\bullet$ over p_0 has been made, there is a canonical smooth π-equivariant map σ from $\widetilde{\Sigma}^\bullet$ to P over the identity mapping of $\Sigma^\bullet$; here π acts on P via ϕ from the right. Explicitly, given a point $\widetilde{p}$ of $\widetilde{\Sigma}^\bullet$, let $\widetilde{w}$ be a smooth path in $\widetilde{\Sigma}^\bullet$ from o to $\widetilde{p}$, write w for its projection into $\Sigma^\bullet$, and let $\widehat{w}$ be the horizontal lift of w, with reference to A and $\widehat{p}_0$; then $\sigma(\widetilde{p})$ equals the end point of $\widehat{w}$. In particular, σ

induces an isomorphism of principal bundles from $\xi_\phi\colon \widetilde{\Sigma}^\bullet \times_\phi G \to \Sigma^\bullet$ to ξ identifying the obvious flat connection A_ϕ on ξ_ϕ with A, but this fact will not be needed below; here $\widetilde{\Sigma}^\bullet \times_\phi G$ is the space arising from $\widetilde{\Sigma}^\bullet \times G$ by identifying points of the kind $(p, \phi(x)b)$ and $(p\phi(x), b)$ as usual.

In Section 4 above, the Lie algebra $\mathfrak{g}$ has been viewed as a *left* π-module via ϕ, written $\mathfrak{g}_\phi$ and, via right translation, we identified the tangent space T_ϕ with the space $C^1(\mathcal{P}, \mathfrak{g}_\phi)$ of $\mathfrak{g}_\phi$-valued *left* 1-cochains on (2.2). We shall use the same notation $\mathfrak{g}_\phi$ for $\mathfrak{g}$, viewed as a *right* π-module. There is *no* conflict of notation since the left and right π-actions on $\mathfrak{g}$ are related by

$$xX = Xx^{-1} = \mathrm{Ad}(\phi(x))X, \quad X \in \mathfrak{g},\ x \in \pi.$$

Accordingly, there are two notions of $\mathfrak{g}_\phi$-valued 1-cocycles: A *left* 1-cocycle is a function u from π to $\mathfrak{g}$ satisfying $u(xy) = u(x) + xu(y)$ $(= u(x) + \mathrm{Ad}(\phi(x))u(y))$ while a *right* 1-cocycle is a function v from π to $\mathfrak{g}$ satisfying $v(xy) = (v(x))y + v(y)$ where $(v(x))y = \mathrm{Ad}(\phi(y)^{-1})(v(x))$. Likewise, via ϕ, the group π acts on the de Rham complex $(\Omega^*(\widetilde{\Sigma}^\bullet, \mathfrak{g}_\phi), d)$ from the right, and we can take invariants $(\Omega^*(\widetilde{\Sigma}^\bullet, \mathfrak{g}_\phi), d)^\pi$. The operator of covariant derivative d_A is a differential on $\Omega^*(\Sigma^\bullet, \mathrm{ad}(\xi))$, and the above map σ induces an isomorphism

$$(\Omega^*(\Sigma^\bullet, \mathrm{ad}(\xi)), d_A) \longrightarrow (\Omega^*(\widetilde{\Sigma}^\bullet, \mathfrak{g}_\phi), d)^\pi \tag{10.1}$$

where on the right-hand side the symbol d refers to the usual de Rham coboundary operator.

Proposition 10.2. *For a flat connection A, with $\phi = \rho(A)$, under the identification of $\mathrm{T}_A(\mathcal{A}(\xi)) = \Omega^1$ with $\Omega^1(\widetilde{\Sigma}^\bullet, \mathfrak{g}_\phi)^\pi$ via (10.1) (in degree 1) and of T_ϕ with $\mathfrak{g}^{2\ell+n} = C^1(\mathcal{P}, \mathfrak{g}_\phi)$ via right translation, the derivative of ρ at A assigns to a closed π-invariant $\mathfrak{g}_\phi$-valued 1-form ϑ on $\widetilde{\Sigma}^\bullet$ the $\mathfrak{g}_\phi$-valued left 1-cocycle u_ϑ for π given by the formula*

$$u_\vartheta(x) = \int_{xo}^{o} \vartheta, \quad \text{for } x \in \pi;$$

here the integral is taken along any smooth path in $\widetilde{\Sigma}^\bullet$ from xo to o, and $\widetilde{\Sigma}^\bullet$ is viewed as a left π-space. This assignment induces the usual isomorphism from $\mathrm{H}^1_A(\Sigma^\bullet, \mathrm{ad}(\xi))$ onto $\mathrm{H}^1(\pi, \mathfrak{g}_\phi)$.

Proof. Theorem 2.7 of [19] entails that, at an arbitrary connection A, not necessarily flat, with

$$\rho(A) = (a_1, b_1, \dots, a_\ell, b_\ell, c_1, \dots, c_n) \in G^{2\ell+n},$$

the differential $d\rho(A)\colon \mathrm{T}_A\mathcal{A}(\xi) \longrightarrow \mathrm{T}_{\rho(A)}G^{2\ell+n}$ of ρ is given by the assignment to $\vartheta \in \Omega^1(\Sigma^\bullet, \mathrm{ad}(\xi)) = \mathrm{T}_A\mathcal{A}(\xi)$ of the vector

$$\left(L_{a_1}\int_{\widehat{x}_1}\vartheta, L_{b_1}\int_{\widehat{y}_1}\vartheta, \dots, L_{a_\ell}\int_{\widehat{x}_\ell}\vartheta, L_{b_\ell}\int_{\widehat{y}_\ell}\vartheta, L_{c_1}\int_{\widehat{z}_1}\vartheta, \dots, L_{c_n}\int_{\widehat{z}_n}\vartheta\right)$$

in $\mathrm{T}_{a_1}G \times \mathrm{T}_{b_1}G \times \dots \times \mathrm{T}_{a_\ell}G \times \mathrm{T}_{b_\ell}G \times \mathrm{T}_{c_1}G \times \dots \times \mathrm{T}_{c_n}G$. However, when A is flat, the tangent map of ρ, combined with the inverse of (10.1) (in degree 1) and with left translation from T_ϕ to $\mathfrak{g}^{2\ell+n}$, looks like

$$\Omega^1(\widetilde{\Sigma}^\bullet, \mathfrak{g}_\phi)^\pi \longrightarrow \Omega^1 \longrightarrow \mathrm{T}_\phi \longrightarrow \mathfrak{g}^{2\ell+n};$$

it is given by the assignment to a π-invariant $\mathfrak g_\phi$-valued 1-form ϑ on $\widetilde\Sigma^\bullet$ of

$$v_\vartheta = \left(\int_{\widehat x_1}\vartheta, \int_{\widehat y_1}\vartheta, \dots, \int_{\widehat x_\ell}\vartheta, \int_{\widehat y_\ell}\vartheta, \int_{\widehat z_1}\vartheta, \dots, \int_{\widehat z_n}\vartheta\right) \in \mathfrak g^{2\ell+n}$$

where, with an abuse of notation, $\widehat x_j$, $\widehat y_j$, and $\widehat z_k$ refer to the unique lifts in $\widetilde\Sigma^\bullet$ with reference to o of, respectively, the closed paths x_j, y_j, and z_k in Σ. This assignment is in fact the degree one twisted integration mapping from $(\Omega^*(\widetilde\Sigma^\bullet, \mathfrak g_\phi), d)^\pi$ to the cellular cochains $(C^*(\Sigma, \mathfrak g_\phi), d)$ with *local coefficients* determined by ϕ, cf. Section 4 of [19] and what is said below. In particular, the cellular 1-cocycles $Z^1(\Sigma, \mathfrak g_\phi)$ with local coefficients coincide with the $\mathfrak g_\phi$-valued right 1-cocycles for π. Thus for a closed 1-form ϑ, the cochain v_ϑ yields a $\mathfrak g_\phi$-valued right 1-cocycle for π. Since integration of a closed 1-form on a simply connected space does not depend on the choice of path but only on the endpoints, this 1-cocycle assigns to $x \in \pi$ the integral

$$v_\vartheta(x) = \int_o^{ox} \vartheta$$

taken along any smooth path from o to ox.

The tangent map of ρ, combined with the inverse of (10.1) (in degree 1) and *right* translation from $\mathrm T_\phi$ to $\mathfrak g^{2\ell+n}$, is given by the assignment to a π-invariant $\mathfrak g_\phi$-valued 1-form ϑ on $\widetilde\Sigma^\bullet$ of the vector

$$\left(\mathrm{Ad}(\phi(x_1))\int_{\widehat x_1}\vartheta, \dots, \mathrm{Ad}(\phi(y_\ell))\int_{\widehat y_\ell}\vartheta, \mathrm{Ad}(\phi(z_1))\int_{\widehat z_1}\vartheta, \dots, \mathrm{Ad}(\phi(z_n))\int_{\widehat z_n}\vartheta\right)$$

in $\mathfrak g^{2\ell+n}$. For a closed 1-form ϑ, this yields the $\mathfrak g_\phi$-valued left 1-cocycle u_ϑ for π assigning the value

$$u_\vartheta(x) = \mathrm{Ad}(\phi(x))v_\vartheta(x) = \mathrm{Ad}(\phi(x))\int_o^{ox}\vartheta = \mathrm{Ad}(\phi(x))\int_o^{x^{-1}o}\vartheta = \int_{xo}^{o}\vartheta$$

to $x \in \pi$. $\square$

Let A be a flat connection on ξ, and let $\phi = \rho(A)$. The right-hand side of (10.1) involves only the vector space of de Rham forms and the homomorphism ϕ but no longer the connection A explicitly. Thus we can completely do away with ξ and the flat connection A and work entirely in terms of ϕ and the local system it defines in the following way, where for the sake of clarity we proceed in somewhat greater generality than actually needed: Let V be a real representation of π; in the application below, V will be $\mathfrak g_\phi$. The de Rham complex $(\Omega^*(\widetilde\Sigma^\bullet, V), d)$ inherits an obvious action of π (whether or not we take the left or right incarnation thereof will not matter any more since both lead to the same result), and we can consider its π-invariants $(\Omega^*(\widetilde\Sigma^\bullet, V), d)^\pi$; by means of an isomorphism of the kind (10.1), the cohomology $\mathrm H^*_{\mathrm{equiv}}(\widetilde\Sigma^\bullet, V)$ of $(\Omega^*(\widetilde\Sigma^\bullet, V), d)^\pi$ actually computes the cohomology of $\Sigma^\bullet$ with values in the corresponding flat vector bundle as usual but this is not important here. Integration

$$(\Omega^*(\widetilde\Sigma^\bullet, V), d)^\pi \to (C^*_{\mathrm{local}}(\Sigma^\bullet, V), d)$$

into the cochains

$$(C^*_{\mathrm{local}}(\Sigma^\bullet, V), d) = (C^*(\widetilde{\Sigma}^\bullet, V), d)^\pi$$

on $\Sigma^\bullet$ with local coefficients determined by V assigns, in particular, to a π-invariant V-valued closed 1-form ϑ on $\widetilde{\Sigma}^\bullet$ the V-valued left 1-cocycle u_ϑ for π given by the formula

$$u_\vartheta(x) = \int_{xo}^{o} \vartheta, \quad \text{for } x \in \pi,$$

and this association induces the standard isomorphism from $\mathrm{H}^1_{\mathrm{local}}(\Sigma^\bullet, V)$ onto $\mathrm{H}^1(\pi, V)$.

Proposition 10.3. *When ϑ is a compactly supported closed π-invariant V-valued 1-form on $\widetilde{\Sigma}^\bullet$, u_ϑ is a parabolic 1-cocycle, that is, for $1 \leq j \leq n$, there is $X_j \in V$ such that*

$$u_\vartheta(z_j) = z_j X_j - X_j.$$

Proof. Let $1 \leq j \leq n$, and pick a point s_j of $\widetilde{\Sigma}^\bullet$. Then

$$\begin{aligned} u_\vartheta(z_j) = \int_{z_j o}^{o} \vartheta &= \int_{z_j o}^{z_j s_j} \vartheta + \int_{z_j s_j}^{s_j} \vartheta + \int_{s_j}^{o} \vartheta \\ &= z_j X_j - X_j + \int_{z_j s_j}^{s_j} \vartheta \end{aligned}$$

where $X_j = \int_o^{s_j} \vartheta$. However, since ϑ is compactly supported it vanishes in a neighborhood of the punctures whence, for a suitable choice of s_j, the integral $\int_{z_j s_j}^{s_j} \vartheta$ is zero. □

Remark 10.4. By pushing the boundary circles further towards the punctures if necessary, in (10.3) above, we can in fact assume that ϑ vanishes on $\Sigma^\bullet \setminus \Sigma$ and hence in particular on the boundary circles $S_1, \ldots, S_n$. For $1 \leq j \leq n$, the point s_j may then be taken to be the end point of the unique lift $\widehat{\gamma}_j$ of γ_j having starting point o so that s_j is a pre-image of p_j and the integral $X_j = \int_o^{s_j} \vartheta$ may be taken along $\widehat{\gamma}_j$ where the notation in Section 2 above is in force. More generally, given *finitely many* compactly supported forms we may still assume that things have been arranged in such a way that these forms vanish on $\Sigma^\bullet \setminus \Sigma$.

Suppose V endowed with a π-invariant symmetric bilinear form $\cdot$; together with the wedge product of forms it induces a bilinear pairing

$$\wedge \colon \Omega^1(\widetilde{\Sigma}^\bullet, V)^\pi \otimes \Omega^1(\widetilde{\Sigma}^\bullet, V)^\pi \longrightarrow \Omega^2(\Sigma^\bullet, \mathbb{R}).$$

We can now spell out the main technical statement of the present section.

Theorem 10.5. *For compactly supported closed π-invariant V-valued 1-forms η and ϑ on $\widetilde{\Sigma}^\bullet$,*

(10.5.1) $$\int_{-\Sigma^\bullet} \eta \wedge \vartheta = \omega_V([u_\eta], [u_\vartheta])$$

where ω_V is the skew-symmetric bilinear pairing (3.6).

Here $-\Sigma^\bullet$ refers to $\Sigma^\bullet$ endowed with the orientation opposite to that determined by the disk D in Section 2 above.

Proof. Write $\widetilde u_\eta$ and $\widetilde u_\vartheta$ for the corresponding groupoid cocycles arising from u_η and u_ϑ by the normalization procedure in step 2 of the proof of Lemma 8.4. In view of (8.6),

$$\omega_V([u_\eta],[u_\vartheta]) = \langle \widetilde c, \widetilde u_\eta \cup \widetilde u_\vartheta\rangle$$

where $\widetilde c$ is the groupoid chain arising from c by the construction in step 3 of the proof of Lemma 8.4.

We may suppose that η and ϑ vanish on $\Sigma^\bullet \setminus \Sigma$, cf. Remark 10.4 above. Then

$$\int_{\Sigma^\bullet} \eta\wedge\vartheta = \int_\Sigma \eta\wedge\vartheta.$$

The cell decomposition of Σ induces a cell decomposition of its universal cover $\widetilde\Sigma$. Extending earlier notation, we denote by $\widehat x_j$, $\widehat y_j$, $\widehat a_k$, and $\widehat \gamma_k$ the unique lifts in $\widetilde\Sigma$ with reference to o of, respectively, the edge paths x_j, y_j, a_k, and γ_k in Σ. These edge paths, together with their left translates under the π-action, constitute the 1-cells of the cell decomposition of $\widetilde\Sigma$. Inspection shows that

$$\widetilde u_\eta(x_j) = -\int_{\widehat x_j}\eta,\quad \widetilde u_\eta(y_j) = -\int_{\widehat y_j}\eta,\quad \widetilde u_\vartheta(x_j) = -\int_{\widehat x_j}\vartheta,\quad \widetilde u_\vartheta(y_j) = -\int_{\widehat y_j}\vartheta,$$
$$\widetilde u_\eta(\gamma_k) = -\int_{\widehat \gamma_k}\eta,\quad \widetilde u_\eta(a_k) = 0,\quad \widetilde u_\vartheta(\gamma_k) = -\int_{\widehat \gamma_k}\vartheta,\quad \widetilde u_\vartheta(a_k) = 0.$$

In other words, the groupoid cocycles $\widetilde u_\eta$ and $\widetilde u_\vartheta$ coincide precisely with the π-equivariant V-valued cellular 1-cocycles $\widehat u_\eta$ and $\widehat u_\vartheta$ arising from η and ϑ, respectively, under the integration mapping

$$(\Omega^*(\widetilde\Sigma, V), d) \longrightarrow (C^*_{\mathrm{cell}}(\widetilde\Sigma, V), d)$$

from V-valued de Rham forms to V-valued cellular cochains $\widetilde\Sigma$, perhaps up to a sign depending on how things have been adjusted but irrelevant for us since the formula (10.5.1) does not depend on this sign; it depends on the orientation of Σ, though.

The disk D mentioned in Section 2 above lifts to a disk $\widehat D$ in $\widetilde\Sigma$, and the left translates of $\widehat D$ under π constitute the 2-cells of $\widetilde\Sigma$. Furthermore,

$$\int_\Sigma \eta\wedge\vartheta = \int_{\widehat D} \eta\wedge\vartheta$$

where on the right-hand side the wedge product $\eta\wedge\vartheta$ is viewed as a 2-form on $\widetilde\Sigma^\bullet$.

The cellular chains of $\widetilde\Sigma$ are given by (2.5). Comparing (2.7) with (8.7), viewing $\widetilde c$ as a groupoid cochain for $\widetilde\pi$, and exploiting the standard fact that the integration

mapping from de Rham cohomology to usual (cellular or singular) cohomology is compatible with multiplicative structures, we conclude that

$$\int_{\widehat D} \eta \wedge \vartheta = -\langle \widetilde c, \widetilde u_\eta \cup \widetilde u_\vartheta \rangle .$$

This completes the proof. A closer look shows that, after a suitable triangulation of Σ arising from the nerve of $\widetilde \pi$, a suitably chosen 2-chain $\widetilde c$ amounts to the negative of the corresponding subdivision of the cellular chain D. For related matters see for example what is said in [28] (IV.5 p. 119) and on p. 495 of [9]. $\square$

Now suppose G compact and connected. Then the Wilson loop mapping ρ identifies the moduli space $N(\xi)$ of gauge equivalence classes of flat connections on ξ with (an open and closed subset of) the representation space $\mathrm{Rep}(\pi, G) = \mathrm{Hom}(\pi, G)\big/G$. Write $N(\xi)_{\mathbf C}$ for the pre-image of the subspace $\mathrm{Rep}(\pi, G)_{\mathbf C} = \mathrm{Hom}(\pi, G)_{\mathbf C}\big/G$ of $\mathrm{Rep}(\pi, G) = \mathrm{Hom}(\pi, G)\big/G$ under this identification; thus $N(\xi)_{\mathbf C}$ consists of gauge equivalence classes of flat connections A so that, for $1 \le j \le n$, the holonomy along some small circle about the j'th puncture lies in the chosen conjugacy class C_j.

Suppose $\cdot$ nondegenerate. For a point $[\phi]$ of the top stratum $\mathrm{Rep}(\pi, G)_{\mathbf C}^{\mathrm{top}}$ of $\mathrm{Rep}(\pi, G)_{\mathbf C}$ (cf. (9.2) above), in view of (4.4) and (4.5), a choice of representative ϕ induces an isomorphism λ_ϕ from $\mathrm H^1_{\mathrm{par}}(\pi, \{\pi_j\}; \mathfrak g_\phi)$ onto the (usual smooth) tangent space $\mathrm T_{[\phi]}(\mathrm{Rep}(\pi, G)_{\mathbf C}^{\mathrm{top}})$; this isomorphism is independent of the choice of ϕ in the sense that, for every $x \in G$, the composite

$$\mathrm H^1_{\mathrm{par}}(\pi, \{\pi_j\}; \mathfrak g_\phi) \xrightarrow{\mathrm{Ad}(x)} \mathrm H^1_{\mathrm{par}}(\pi, \{\pi_j\}; \mathfrak g_{x\phi}) \xrightarrow{\lambda_{x\phi}} \mathrm T_{[\phi]}(\mathrm{Rep}(\pi, G)_{\mathbf C}^{\mathrm{top}})$$

coincides with λ_ϕ. This makes precise the folklore statement that 'the tangent space is the first cohomology group with coefficients in the corresponding Lie algebra representation'; details for the special case with no punctures have been worked out in Section 7 of [19].

Under the Wilson loop mapping, the top stratum $\mathrm{Rep}(\pi, G)_{\mathbf C}^{\mathrm{top}}$ of $\mathrm{Rep}(\pi, G)_{\mathbf C}$ corresponds to the subspace $N(\xi)_{\mathbf C}^{\mathrm{top}}$ of points $[A]$ of $N(\xi)_{\mathbf C}$ whose representatives A have minimal stabilizer subgroup (in the group of gauge transformations). For a flat connection A, write $\mathrm H^1_{A,c}(\Sigma^\bullet, \mathrm{ad}(\xi))$ for the subgroup of the first cohomology group $\mathrm H^1_A(\Sigma^\bullet, \mathrm{ad}(\xi))$ generated by classes of compactly supported 1-forms. A choice of representative A of a point $[A]$ of $N(\xi)_{\mathbf C}^{\mathrm{top}}$ induces an isomorphism λ_A from $\mathrm H^1_{A,c}(\Sigma^\bullet, \mathrm{ad}(\xi))$ onto the (usual smooth) tangent space $\mathrm T_{[A]}(N(\xi)_{\mathbf C}^{\mathrm{top}})$ and, for a gauge transformation ν, the composite

$$\mathrm H^1_{A,c}(\Sigma^\bullet, \mathrm{ad}(\xi)) \xrightarrow{\nu} \mathrm H^1_{\nu A,c}(\Sigma^\bullet, \mathrm{ad}(\xi)) \xrightarrow{\lambda_{\nu A}} \mathrm T_{[A]}(N(\xi)_{\mathbf C}^{\mathrm{top}})$$

coincides with λ_A.

Let A be a flat connection on ξ representing a point of $N(\xi)_{\mathbf C}$ and let $\phi = \rho(A)$. The isomorphism (10.1) identifies $\mathrm H^1_{A,c}(\Sigma^\bullet, \mathrm{ad}(\xi))$ with the subgroup $\mathrm H^1_{\mathrm{equiv},c}(\widetilde\Sigma^\bullet, \mathfrak g_\phi)$ of $\mathrm H^1_{\mathrm{equiv}}(\widetilde\Sigma^\bullet, \mathfrak g_\phi)$ generated by classes of compactly supported closed equivariant 1-forms. In view of (10.3), integration identifies $\mathrm H^1_{\mathrm{equiv},c}(\widetilde\Sigma^\bullet, \mathfrak g_\phi)$ with $\mathrm H^1_{\mathrm{par}}(\pi, \{\pi_j\}; \mathfrak g_\phi)$.

Suppose in addition that $[A]$ lies in $N(\xi)^{\mathrm{top}}_{\mathbf{C}}$. Up to signe, the gauge theory description of the 2-form on $N(\xi)^{\mathrm{top}}_{\mathbf{C}}$ induced by $\cdot$ is given on the tangent space $\mathrm{T}_{[A]}(N(\xi)^{\mathrm{top}}_{\mathbf{C}})$ by the left-hand side of (10.5.1) with $V = \mathfrak{g}_\phi$. On the tangent space $\mathrm{T}_\phi\mathrm{Rep}(\pi, G)^{\mathrm{top}}_{\mathbf{C}}$, via right translation, the 2-form induced by $\cdot$ is given on $\mathrm{H}^1_{\mathrm{par}}(\pi, \{\pi_j\}; \mathfrak{g}_\phi)$ by the right-hand side of (10.5.1) with $V = \mathfrak{g}_\phi$. Theorem 10.5 implies at once that the two forms correspond, up to sign. This identifies the gauge theory description of the symplectic form with the representation space description given in the present paper. Notice that the given identification is independent of the symplecticity.

A similar statement can be made at an arbitrary point $[A]$ of $N(\xi)_{\mathbf{C}}$ and the corresponding point $[\phi]$ of $\mathrm{Rep}(\pi, G)_{\mathbf{C}}$ where $\phi = \rho(A)$. Choices of ϕ and A still determine linear maps of the kind λ_A and λ_ϕ but these maps will in general no longer be isomorphisms. More precisely, λ_ϕ induces an isomorphism of the subspace $\mathrm{H}^1_{\mathrm{par}}(\pi, \{\pi_j\}; \mathfrak{g}_\phi)^{Z_\phi}$ of invariants onto the smooth tangent space at $[\phi]$ of the stratum in which $[\phi]$ lies where $Z_\phi \subseteq G$ refers to the stabilizer of ϕ; a corresponding statement can be made for A. This has been worked out for the closed case (no punctures) in Section 7 of [19].

When G is not compact, while the statement of (10.5) is still available, there is no good space of gauge equivalence classes of flat connections nor is there a good space of representations since there are orbits which are not closed. The appropriate generalization of the present results should involve certain categorical or algebraic quotients.

References

1. M. Atiyah, *The geometry and physics of knots*, Cambridge University Press, Cambridge, U. K., 1990.
2. M. Atiyah and R. Bott, *The Yang-Mills equations over Riemann surfaces*, Phil. Trans. R. Soc. London A **308** (1982), 523–615.
3. R. Bieri and B. Eckmann, *Relative homology and Poincaré duality for group pairs*, J. Pure and Applied Algebra **13** (1978), 277–319.
4. I. Biswas and K. Guruprasad, *Principal bundles on open surfaces and invariant functions on Lie groups*, Int. J. of Math. **4** (1993), 535–544.
5. I. Biswas and K. Guruprasad, *On some geometric invariants associated to the space of flat connections*, Can. J. of Math. (to appear).
6. R. Bott, *On the Chern-Weil homomorphism and the continuous cohomology of Lie groups*, Advances **11** (1973), 289–303.
7. R. Bott, H. Shulman, and J. Stasheff, *On the de Rham theory of certain classifying spaces*, Advances **20** (1976), 43–56.
8. R. Brown, *Elements of modern topology*, McGraw-Hill, London, 1968.
9. S. Eilenberg and S. Mac Lane, *Relations between homology and homotopy groups of spaces*, Ann. of Math. **46** (1945), 480–509.
10. V. V. Fock and A. A. Rosly, *Flat connections and polyubles*, in: Modern problems in quantum field theory, strings, and quantum gravity, Kiev 1992, Teor. Mat. Fiz. **95** (1993), 228–238.
11. W. M. Goldman, *The symplectic nature of fundamental groups of surfaces*, Advances **54** (1984), 200–225.

12. K. Guruprasad, *Flat connections, geometric invariants, and the symplectic nature of the fundamental group of surfaces*, Pacific J. of Math. **162** (1984), 45–55.
13. K. Guruprasad and C. S. Rajan, *Group cohomology and the symplectic structure on the moduli space of representations*, preprint, McGill University, 1995.
14. S. Helgason, *Differential geometry, Lie groups, and symmetric spaces*, Pure and Applied Mathematics, vol. 80, Academic Press, New York · San Francisco · London, 1978.
15. J. Huebschmann, *Symplectic and Poisson structures of certain moduli spaces*, hep-th 9312112, Duke Math. J. (to appear).
16. J. Huebschmann, *Symplectic and Poisson structures of certain moduli spaces. II. Projective representations of cocompact planar discrete groups*, dg-ga/9412003, Duke Math. J. (to appear).
17. J. Huebschmann, *The singularities of Yang-Mills connections for bundles on a surface. I. The local model*, Math. Z., to appear, dg-ga/9411006.
18. J. Huebschmann, *The singularities of Yang-Mills connections for bundles on a surface. II. The stratification*, Math. Z., to appear, dg-ga/9411007.
19. J. Huebschmann, *Smooth structures on moduli spaces of central Yang-Mills connections for bundles on a surface*, preprint 1992, dg-ga/9411008.
20. J. Huebschmann, *Poisson structures on certain moduli spaces for bundles on a surface*, Annales de l'Institut Fourier **45** (1995), 65–91.
21. J. Huebschmann, *Poisson geometry of flat connections for* SU(2)*-bundles on surfaces*, Math. Z., to appear, hep-th/9312113.
22. J. Huebschmann, *Poisson geometry of certain moduli spaces*, Lectures delivered at the 14th winter school, Czech Republic, Srni, January 1994, Rendiconti del circolo matematico di Palermo, to appear.
23. J. Huebschmann and L. Jeffrey, *Group cohomology construction of symplectic forms on certain moduli spaces*, Int. Math. Research Notices **6** (1994), 245–249.
24. L. Jeffrey, *Extended moduli spaces of flat connections on Riemann surfaces*, Math. Ann. **298** (1994), 667–692.
25. L. Jeffrey, *Symplectic forms on moduli spaces of flat connections on 2-manifolds*, preprint alg-geom/9411008, to appear in *Proceedings of the Georgia International Topology Conference*, Athens, Ga. 1993, ed. by W. Kazez.
26. L. Jeffrey, *Group cohomology construction of the cohomology of moduli spaces of flat connections on 2-manifolds*, Duke Math. J. **77** (1995), 407–429.
27. Y. Karshon, *An algebraic proof for the symplectic structure of moduli space*, Proc. Amer. Math. Soc. **116** (1992), 591–605.
28. S. Mac Lane, *Homology*, Die Grundlehren der mathematischen Wissenschaften No. 114, Springer, Berlin · Göttingen · Heidelberg, 1963.
29. M. S. Narasimhan and C. S. Seshadri, *Stable and unitary vector bundles on a compact Riemann surface*, Ann. of Math. **82** (1965), 540–567.
30. H. B. Shulman, *Characteristic classes and foliations*, Ph. D. Thesis, University of California, 1972.
31. R. Sjamaar and E. Lerman, *Stratified symplectic spaces and reduction*, Ann. of Math. **134** (1991), 375–422.
32. H. F. Trotter, *Homology of group systems with applications to knot theory*, Annals of Math. **76** (1962), 464–498.

33. A. Weil, *Remarks on the cohomology of groups*, Ann. of Math. **80** (1964), 149–157.
34. A. Weinstein, *The symplectic structure on moduli space*, The A. Floer memorial volume, H. Hofer, C. Taubes, A. Weinstein, and E. Zehnder, eds., Birkhäuser Verlag, Basel · Boston · Berlin, 1995, pp. 627–635.